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Analysis and treatment impact to control the transmission dynamics of skin sores disease with novel hybrid fractional operators

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Abstract: Skin sores (impetigo) are a frequent and contagious skin infection in young children, causing blisters and ulcers. It is usually not hazardous and resolves within a week of treatment or a few weeks if no treatment is used. Treatment is commonly recommended since it can shorten the length of the condition and reduce the risk of the virus spreading to others. We developed and tested a non-linear hybrid fractional order model to investigate skin sores infection and the transmission dynamics. The study examines the mathematical properties of the suggested model, such as the feasible region, equilibrium points, basic reproduction number, and existence of the system's unique solutions using Banach fixed-point theory. It also uses the appropriate lyapunov function to examine the stability of equilibrium states. This work gives a thorough examination of several hybrid fractional operators and numerically simulates the suggested skin sores system using the Laplace-Adomian decomposition approach, demonstrating its efficacy in simulating theoretical scenarios. This research advances our knowledge of the mechanisms underlying disease transmission by taking into account fractional-order dynamics and a variety of routes, offering suggestions for better disease management and control.

Keywords: Constant Proportional(CP) operator, Impetigo (skin sores); Strength number; Hilfer Generalized Proportional

1. Introduction

Impetigo (or Skin Sores) is the most prevalent and contagious skin infection that usually afflicts infants and young children. Bacteria that infect the outer layers of the pores and skin, generally Staphylococcus aureus or Streptococcus pyogenic, are the cause of impetigo[1]. It usually starts as red, itchy sores that scab over and, for some days, ooze clean fluid or pus. The sore then starts to crust up and turn yellow or "honey-colored," and it has treatment options without leaving a scar. Impetigo can have an effect on the skin everywhere on the body, although it most often affects the vicinity across the lips, fingers, and forearms, as well as the diaper vicinity in small kids. Impetigo comes in three varieties: The most common form of impetigo in adults is non-bullous impetigo. Thick crusts with a honey tint are the result. Bullous impetigo results in significant skin blistering. Ecthyma, a particularly severe kind that regularly results from impetigo that is left untreated, results in ulcerative infections that penetrate the deeper skin layers [2]. Impetigo can affect everyone, but it most often affects youngsters, specifically those between the ages of 2 and 14. Impetigo is more common and regularly occurs in underdeveloped countries and in sections of industrialised international locations with lower earnings. In step with the latest estimates, impetigo currently affects everywhere between 111 million kids in growing nations and 140 million human beings globally [3].

Figure 1: Skin Sores or blisters can develop anywhere, but are most common in exposed areas.

Mathematical modeling and evaluation are crucial to the study of infectious contamination epidemiology. In their discussion of the potential of mathematical modeling to improve our comprehension of skin illness, Tanaka and Ono [4] emphasized the necessity of close ties between statistical analysis, mathematical modeling, and experiments in order to improve skin research in the postgenomic era. Anissimov [5] assessed the suitability of two kinds of mathematical models for assessing the toxicity of pores and skin. The mathematical models developed by Nakaoka et al. [6] provide a thorough framework to describe the interplay between bacterial species as an environmental element and host immune responses at the dermis. To determine the strength of exposure and productive period of skin sores in isolated Australian communities, Lydeamore et al. [7] created a stochastic variant of the SIS model. Saidalieva et al. [8] used well-diagnosed data and theoretical viewpoints from the disciplines of biology, biophysics, and law to examine the regulatory mechanisms and dynamics of skin cancer. Zhao et al. [9] proposed a co-contamination of Buruli ulcer and cholera to analyze the most efficient strategy to containing the spread of both illnesses using a mathematical model that safeguarded five capability controls. In their mathematical model, Greugny et al. [10] used opportunistic infections and pores and skin commensals to examine the mechanisms driving one species' dominance over the other. A mathematical model of the SIRS type was created by Parvin et al. [11] to illustrate the effects of UV light on pores and skin cancer.

The idea of fractional derivatives is introduced in fractional calculus, an extension of classical calculus that greatly improves our understanding of a variety of phenomena [12, 13, 14, 15]. Because fractional calculus, as opposed to integer order approaches, may include memory and heredity functions in epidemic frameworks, it is becoming more and more popular. Since many students have created non-integer models for different diseases, showcasing its potential for modeling complicated systems [16, 17, 18, 19], this method is essential for comprehending how diseases propagate. In [20], the most popular maximum-order fractional-order differential operators were applied in three exceptional ways to examine an epidemiological version related to the chickenpox, a contaminant typically found in youngsters. In [21], a fractional-order model was developed to comprehend the transmission and management of foot-and-mouth disease. It covers quarantine for diseased animals and immunization plans for vulnerable animals. The model, which was validated using MATLAB, shows that while animal isolation and vaccination are essential for managing the disease at specific transmission levels, they might not be enough if transmission above a threshold. Using Atangana-Baleanu fractional derivatives, Samuel Okyere and Joseph Ackora-Prah [22] investigated the transmission dynamics of monkeypox and found that these derivatives had a major impact on community dynamics. In order to investigate mumps-induced hearing loss in children, Nisar et al. [23] created a hybrid fractional order system that performs better than the integer-order derivative while retaining the system memory effect. Using the Caputo-fractional derivative, P. Bedi et al. [24] created a nonlinear fractional-order model to examine the dynamic behavior of vector-borne illnesses. To demonstrate their theoretical conclusions, they performed numerical simulations and contrasted the outcomes with integer-order derivative results. In [25], an immune system-boosting fractional order mathematical model was developed to investigate how hand contamination may cause pink eye infection and how early vaccinations can treat it. A mathematical model for human immunodeficiency virus type 1 infection in CD4+ T-cells was created by researchers in [26]. It included fractional-order dynamics and graphical representations to show how model parameters affected the infection. The study [27] created a Caputo fractional order model for the control of pneumonia infections and used numerical simulation to examine how it affected specific model parameters. Baleanu et al. [28] suggested a more versatile and generalized operator, the constant-proportional Caputo fractional derivative. Furthermore, the proportional derivative was merged with two well-known fractional derivatives by Ali Akgül [29]. As can be seen in [30, 31, 32, 33], researchers are using these definitions to produce a number of useful results.

2. Fundamentals

Definition 2.1. In [28], Baleanu et al. created two hybrid fractional operators known as Proportional Caputo (PC) and constant proportional Caputo (CPC) as given below:

$$\begin{cases} {}_0^{\mathrm{PC}}\mathrm{D}_t^{\mu}\rho(t) = \frac{1}{\Gamma(1-\mu)}\int_0^t \Big(\mathcal{A}_1(\mu,\varepsilon)\rho(\varepsilon)+\mathcal{A}_0(\mu,\varepsilon)\rho'(\varepsilon)\Big)(t-\varepsilon)^{-\mu}d\varepsilon \\ \qquad\qquad\quad = {}_0^{RL}\mathrm{I}_t^{1-\mu}\Big[\mathcal{A}_1(\mu,t)\rho(t)+\mathcal{A}_0(\mu,t)\rho'(t)\Big] \\ {}_0^{CPC}\mathrm{D}_t^{\mu}\rho(t) = \frac{1}{\Gamma(1-\mu)}\int_0^t \left[\mathcal{A}_1(\mu)\rho(\varepsilon)+\mathcal{A}_0(\mu)\rho'(\varepsilon)\right](t-\varepsilon)^{-\mu}d\varepsilon \\ \qquad\qquad\quad = \mathcal{A}_1(\mu)\,{}_0^{RL}I_t^{1-\mu}\rho(t)+\mathcal{A}_0(\mu)\,{}_0^{C}\mathrm{D}_t^{\mu}\rho(t). \end{cases} \tag{1}$$

Definition 2.2. Ali Akgül [29] suggested two new hybrid fractional operators, Constant-Proportional Atangana-Baleanu (CPABC) and Constant-Proportional Caputo-Fabrizio (CPCF) operators, as stated below:

$$\begin{cases} {}_0^{CPABC}\mathsf{D}_t^{\mu}\rho(t) = \frac{\mathbb{AB}(\mu)}{1-\mu}\int_0^t \left[\mathcal{A}_1(\mu)\rho(\varepsilon)+\mathcal{A}_0(\mu)\rho'(\varepsilon)\right]E_{\mu}(-\frac{\mu}{1-\mu}(t-\varepsilon))d\varepsilon \\ \qquad = \frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\int_0^t \rho(\varepsilon)E_{\mu}(-\frac{\mu}{1-\mu}(t-\varepsilon)^{\mu})d\varepsilon + \frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\int_0^t \rho'(\varepsilon)E_{\mu}(-\frac{\mu}{1-\mu}(t-\varepsilon)^{\mu})d\varepsilon \\ \qquad = \rho(t)\cdot\frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)}{1-\mu}E_{\mu}(-\frac{\mu}{1-\mu}t^{\mu}) + \rho'(t)\cdot\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}{1-\mu}E_{\mu}(-\frac{\mu}{1-\mu}t^{\mu}). \end{cases} \tag{2}$$

$$\begin{cases} {}_0^{CPCF}\mathsf{D}_t^{\mu}\rho(t) = \frac{\mathbb{M}(\mu)}{1-\mu}\int_0^t \left[\mathcal{A}_1(\mu)\rho(\varepsilon)+\mathcal{A}_0(\mu)\rho'(\varepsilon)\right]\exp(-\frac{\mu}{1-\mu}(t-\varepsilon))d\varepsilon \\ \qquad = \frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\int_0^t \rho(\varepsilon)\exp(-\frac{\mu}{1-\mu}(t-\varepsilon))d\varepsilon + \frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\int_0^t \rho'(\varepsilon)\exp(-\frac{\mu}{1-\mu}(t-\varepsilon))d\varepsilon \\ \qquad = \rho(t)\cdot\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\exp(-\frac{\mu}{1-\mu}t) + \rho'(t)\cdot\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\exp(-\frac{\mu}{1-\mu}t). \end{cases} \tag{3}$$

3. Mathematical Model with CPC operator

Here, we address a hybrid edition of the transmission dynamics of skin sores model that has time-fractional order [3]. In our model, we divide the population into three categories: susceptible, infected, and recovered, denoted by $\mathbb{J}$, $\mathbb{K}$, and $\mathbb{L}$, respectively. "β" stands for the rate of recruitment into the susceptible class, "δ" for the effective contact rate with infected people, "λ" for the rate of recovery, "η" for the rate of susceptibility of those who have recovered, and "ϕ" for the rate of natural death. In a set of fractional differential equations given below, the parameters should be positive and biologically impacted.

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}(t) = \beta - \delta\frac{\mathbb{JK}}{\mathbb{N}} + \eta\mathbb{L} - \phi\mathbb{J}, \\ {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}(t) = \delta\frac{\mathbb{JK}}{\mathbb{N}} - (\lambda+\phi)\mathbb{K}, \\ {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}(t) = \lambda\mathbb{K} - (\eta+\phi)\mathbb{L}. \end{cases} \tag{4}$$

with non-negative initial constraints,

$$\mathbb{J}(0) = \mathbb{J}^0 \ , \ \mathbb{K}(0) = \mathbb{K}^0 \ , \ \mathbb{L}(0) = \mathbb{L}^0 \tag{5}$$

3.1. Positivity, Boundedness and Biological Feasibility

We begin with class $\mathbb{L}(t)$:

$${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}(t) = \lambda\mathbb{K} - (\eta+\phi)\mathbb{L} \geq -(\eta+\phi)\mathbb{L}, \quad \forall t>0 \quad \Longrightarrow \mathbb{L}(t) \geq \mathbb{L}^0 e^{-(\eta+\phi)t}, \quad \forall t>0, \tag{6}$$

for classes $\mathbb{J}(t)$ and $\mathbb{K}(t)$, we require to define the norm:

$$\|\zeta\|_\infty = \sup_{t\in D_\zeta}|\zeta(t)|, \tag{7}$$

while D_ζ is the domain of ζ. Utilizing this norm, we have for the class $\mathbb{J}(t)$;

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}(t) = \quad \beta - \delta\frac{\mathbb{JK}}{\mathbb{N}} + \eta\mathbb{L} - \phi\mathbb{J} \ \geq -\delta\frac{\mathbb{JK}}{\mathbb{N}} - \phi\mathbb{J} = -(\delta\frac{|\mathbb{K}|}{|\mathbb{N}|}+\phi)\mathbb{J} \quad \forall t>0 \\ \qquad \geq -(\delta\frac{\sup_{\approx\in\mathbb{D}_\mathbb{K}}|\mathbb{K}|}{\sup_{\approx\in\mathbb{D}_\mathbb{N}}|\mathbb{N}|}+\phi)\mathbb{J} = -(\delta\frac{|\mathbb{K}|_\infty}{|\mathbb{N}|_\infty}+\phi)\mathbb{J}, \quad \forall t>0 \\ \qquad \Rightarrow \mathbb{J}(t) \geq \mathbb{J}^0 e^{-(\delta\frac{|\mathbb{K}|_\infty}{|\mathbb{N}|_\infty}+\phi)t}, \quad \forall t>0. \end{cases} \tag{8}$$

Similarly, we find

$$\mathbb{K}(t) \geq \mathbb{K}^0 e^{-(\lambda+\phi-\delta\frac{|\mathbb{J}|_\infty}{|\mathbb{N}|_\infty})t}, \quad \forall t>0. \tag{9}$$

We also find

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{N}(t) = {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}(t) + {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}(t) + {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}(t) \\ \qquad = \beta - \phi(\mathbb{J}+\mathbb{K}+\mathbb{L}) = \beta - \phi\mathbb{N}. \end{cases} \tag{10}$$

It implies that ${}_{0}^{CPC}\mathrm{D}_t^{\mu}\mathbb{N}(t) \leq 0$ if $\mathbb{N} \geq \frac{\beta}{\phi}$. Specially $\mathbb{N}(t) \leq \frac{\beta}{\phi}$ if $\mathbb{N}(0) \leq \frac{\beta}{\phi}$ Therefore, the feasible region is as:

$$\Pi = \{(\mathbb{J},\mathbb{K},\mathbb{L}) \in \mathbf{R}_+^3 : \mathbb{N} \leq \frac{\beta}{\phi}\}. \tag{11}$$

3.2. Existence and uniqueness of solutions of the proposed model

Let

$$\begin{cases} \lambda_1(t,\mathbb{J},\mathbb{K},\mathbb{L}) = \beta - \delta\frac{\mathbb{J}\mathbb{K}}{\mathbb{N}} + \eta\mathbb{L} - \phi\mathbb{J}, \\ \lambda_2(t,\mathbb{J},\mathbb{K},\mathbb{L}) = \delta\frac{\mathbb{J}\mathbb{K}}{\mathbb{N}} - (\lambda+\phi)\mathbb{K}, \\ \lambda_3(t,\mathbb{J},\mathbb{K},\mathbb{L}) = \lambda\mathbb{K} - (\eta+\phi)\mathbb{L}. \end{cases} \tag{12}$$

Define a Banach space $\mathbf{S}[0,\mathrm{T}] = \mathrm{B}$ under the norm

$$\|\Xi\| = \sup_{t\in[0,\mathrm{T}]} \big(|\mathbb{J}(t)| + |\mathbb{K}(t)| + |\mathbb{L}(t)|\big). \tag{13}$$

$$\textit{where} \quad \vartheta(t) = \begin{cases} \mathbb{J}(t) \\ \mathbb{K}(t) \\ \mathbb{L}(t) \end{cases} \quad \vartheta_0(t) = \begin{cases} \mathbb{J}^0 \\ \mathbb{K}^0 \\ \mathbb{L}^0 \end{cases} = \begin{cases} \lambda_1(t,\mathbb{J},\mathbb{K},\mathbb{L}), \\ \lambda_2(t,\mathbb{J},\mathbb{K},\mathbb{L}), \\ \lambda_3(t,\mathbb{J},\mathbb{K},\mathbb{L}). \end{cases} \tag{14}$$

From above set, we can express the system (4) as

$$\begin{cases} {}_{0}^{CPC}\mathrm{D}_0^{\mu}\vartheta(t) = \omega(t,\vartheta(t)), \ \ t \in [0,\mathrm{T}], \\ \vartheta(0) = \vartheta_0. \end{cases} \tag{15}$$

(15) is equivalent to Volterra integral equation [34]:

$$\vartheta(t) = \vartheta(0)\exp\big(-\int_0^t \frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big) + \frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s \exp\big(-\int_s^t \frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\frac{\omega(\varepsilon,\vartheta(\varepsilon))(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}d\varepsilon ds, \tag{16}$$

where $t \in [0,\mathrm{T}]$. we make hypothesis as stated below:
(i) : $|\omega(t,\vartheta(t))| \leq G_U|\vartheta|^p + G_V$
(ii): For every $\vartheta, \overline{\vartheta}$ there exists a constant $G_L > 0$ such that $|\omega(t,\vartheta) - \omega(t,\overline{\vartheta})| \leq G_L\|\vartheta - \overline{\vartheta}\|$
We define an operator $\kappa : \mathrm{B} \longrightarrow \mathrm{B}$ as:

$$\kappa\vartheta(t) = \vartheta_0\exp\big(-\int_0^t \frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big) + \frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s \exp\big(-\int_s^t \frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\omega(\varepsilon,\vartheta(\varepsilon))d\varepsilon ds. \tag{17}$$

Theorem 3.1. Assume that if hypotheses (i) and (ii) are correct, then equation (15) contains at least one solution, which is required for its justification.

Proof. **(1).** Let's start by assuming that κ is continuous. Assume that $\omega(\varepsilon,\vartheta(\varepsilon))$ is also continuous as ϑ is a continuous. Moreover, if for $\vartheta,\ \vartheta_p \in \mathbf{B} \ \exists\ \vartheta_p \longrightarrow \vartheta$ then we get $\kappa\vartheta_p \longrightarrow \kappa\vartheta$. Consider

$$\begin{cases} \|\kappa\vartheta_p - \kappa\vartheta\| = \max_{t\in[0,\mathrm{T}]}\Big|\frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s \exp\big(-\int_s^t \frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\big[\omega_p(\varepsilon,\vartheta_p(\varepsilon)) - (\omega(\varepsilon,\vartheta(\varepsilon))\big]d\varepsilon ds\Big| \\ \qquad \leq \max_{t\in[0,\mathrm{T}]}\frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s\Big|\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\Big|\Big|\omega_p(\varepsilon,\vartheta_p(\varepsilon)) - (\omega(\varepsilon,\vartheta(\varepsilon))\Big|d\varepsilon ds \ \leq \frac{\mathrm{T}^{\mu}}{\Gamma(\mu+1)}G_L\|\vartheta_p - \vartheta\|. \end{cases} \tag{18}$$

$\frac{\mathrm{T}^{\mu}}{\Gamma(\mu+1)}G_L\|\vartheta_p-\vartheta\|\longrightarrow 0$ as $p\longrightarrow\infty$. As ω is continuous κ also continuous.
(2). We shall now demonstrate that κ is bounded. Assume that κ satisfies the growth requirement for this purpose.

$$\begin{cases}\|\kappa\vartheta\|=\max_{t\in[0,\mathrm{T}]}\left|\vartheta_0\exp(-\int_0^t\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu)+\frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s\exp(-\int_s^t\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu)\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\omega(\varepsilon,\vartheta(\varepsilon))d\varepsilon ds\right|\\ \quad\leq\left|\vartheta_0\exp\big(-\int_0^t\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\right|+\left|\frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s\exp\big(-\int_s^t\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\times\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\omega(\varepsilon,\vartheta(\varepsilon))d\varepsilon ds\right|\\ \quad\leq\left|\vartheta_0\right|+\max_{t\in[0,\mathrm{T}]}\frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s\left|\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\right|\left|\omega(\varepsilon,\vartheta(\varepsilon))\right|d\varepsilon ds\leq\left|\vartheta_0\right|+\frac{\mathrm{T}^{\mu}}{\Gamma(\mu+1)}\Big(G_U|\vartheta|^p+G_V\Big).\end{cases}\tag{19}$$

This implies that κ is bounded.
(3). Here, we demonstrate the equicontinuity of κ. Assume $t_1,t_2\in[0,\mathrm{T}]$ for this reason. For $t_2<t_1$ and $\vartheta\in\mathscr{F}_r$, we determine that

$$\begin{cases}|\kappa\vartheta(t_1)-\kappa\vartheta(t_2)|\leq\left|\vartheta_0\exp\big(-\int_0^{t_1}\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)-\vartheta_0\exp\big(-\int_0^{t_2}\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\right|\\ \quad+\frac{1}{\Gamma(\mu-1)}\left|\int_0^{t_1}\int_0^s\Big(\exp\big(-\int_0^{t_1}\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)-\exp\big(-\int_s^{t_2}\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\Big)\times\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\omega(\varepsilon,\vartheta(\varepsilon))d\varepsilon ds\right|\\ \quad+\frac{1}{\Gamma(\mu-1)}\left|\int_{t_2}^{t_1}\int_0^s\exp\big(-\int_0^{t_2}\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\omega(\varepsilon,\vartheta(\varepsilon))d\varepsilon ds\right|\\ \quad\leq\Big(\left|\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\vartheta_0\right|+\frac{G_U|\vartheta|^p+G_V}{\Gamma(\mu+1)}\left|\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0^2(\mu)}\right|t_1\Big)(t_1-t_2)+\frac{G_U|\vartheta|^p+G_V}{\Gamma(\mu+1)}\left|\frac{1}{\mathcal{A}_0(\mu)}\right|(t_1^{\mu}-t_2^{\mu}),\end{cases}\tag{20}$$

where $\eta\in(t_1,t_2)$. Since $t_2\longrightarrow t_1$, the RHS of above inequality approaches to zero independently of $\vartheta\in\mathscr{F}_r$. According to the Arzela-Ascoli theorem, it is hence compact. **(4).** Lastly, we demonstrate that the set below is bounded.

$$\mathbf{W}(\kappa)=\{\vartheta\in\mathbf{B}:\vartheta=w\kappa\vartheta,w\in(0,1)\}.\tag{21}$$

For this, let $\forall\,t\in[0,\mathrm{T}]$ and $\vartheta\in\mathbf{W}(\kappa)$, then

$$|\vartheta(t)|=w|\vartheta(t)|\leq w\Big(|\vartheta_0|+\frac{\mathrm{T}^{\mu}}{\Gamma(\mu+1)}\big[G_U|\vartheta|^p+G_V\big]\Big).\tag{22}$$

Since it is bounded, κ has at least one solution in accordance with Schaefer's fixed-point theorem. □

Theorem 3.2. If $\frac{\mathrm{T}^{\mu}G_U}{\Gamma(\mu+1)}<1$ then equation (15) has unique solution.

Proof. Considеer $\vartheta,\overline{\vartheta}\in\mathscr{D}$, then

$$\begin{cases}\|\kappa\vartheta-\kappa\overline{\vartheta}\|=\max_{t\in[0,\mathrm{T}]}\left|\frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s\exp\big(-\int_s^t\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}d\nu\big)\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\big[\omega(\varepsilon,\vartheta(\varepsilon))-(\omega(\varepsilon,\overline{\vartheta}(\varepsilon))\big]d\varepsilon ds\right|\\ \quad\leq\max_{t\in[0,\mathrm{T}]}\frac{1}{\Gamma(\mu-1)}\int_0^t\int_0^s\left|\frac{(s-\varepsilon)^{\mu-2}}{\mathcal{A}_0(\mu)}\right|\left|\omega(\varepsilon,\vartheta(\varepsilon))-(\omega(\varepsilon,\overline{\vartheta}(\varepsilon))\right|d\varepsilon ds\leq\frac{\mathrm{T}^{\mu}}{\Gamma(\mu+1)}G_L\|\vartheta-\overline{\vartheta}\|.\end{cases}\tag{23}$$

Therefore, it is unique and has specific solution. □

3.3. Equilibrium Points

Disease-free equilibrium occurs in the absence of disease. Hence, We find the disease-free equilibrium states ($\mathbf{E}^0$) as given below:

$$\mathcal{E}_0=(\mathsf{J}_0,\mathsf{K}_0,\mathsf{L}_0)=\Big(\frac{\beta}{\phi},0,0\Big).\tag{24}$$

Endemic equilibrium point, denoted by $\mathcal{E}^{\star}$, is as stated below:

$$\mathcal{E}^{\star}=\begin{cases}\mathsf{J}^{\star}=\frac{(\lambda+\phi)\mathbb{N}}{\delta}, \qquad \mathsf{K}^{\star}=-\frac{(\phi+\eta)\left[\mathbb{N}\phi(\lambda+\phi)-\beta\delta\right]}{\phi\delta(\lambda+\phi+\eta)},\\ \mathsf{L}^{\star}=-\frac{\lambda\left[\mathbb{N}\phi(\lambda+\phi)-\beta\delta\right]}{\phi\delta(\lambda+\phi+\eta)}.\end{cases}\tag{25}$$

3.4. Reproductive Number

On system (4), we apply the next generation matrix approach [35]. The reproductive number $\mathcal{R}_0$ is as follows:

$$\mathcal{R}_0 = \frac{\delta}{\lambda + \phi}. \tag{26}$$

3.5. Stability Analysis

Lemma 3.3. *[36] Assume that* $\mathtt{F}(t) \in \mathbf{R}^+$ *be a continuous function and for any* $t \geq t_0$*;*

$$ {}_0^{CPC}\mathsf{D}_t^{\mu}\left(\mathtt{F}(t) - \mathtt{F}^\star - \mathtt{F}^\star \ln \frac{\mathtt{F}(t)}{\mathtt{F}^\star}\right) \leq \left(1 - \frac{\mathtt{F}^\star}{\mathtt{F}(t)}\right){}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}(t) \quad , \quad \mathtt{F}^\star \in \mathbf{R}^+ \, , \forall \mu \in (0,1). \tag{27}$$

Theorem 3.4. *If* $\mathcal{R}_0 < 1$*, then* $\mathcal{E}_0$ *is globally asymptotically stable.*

Proof. Define Lyapunov function as

$$\mathtt{F} = (\mathbb{J} - \mathbb{J}_0 - \mathbb{J}_0 \ln \frac{\mathbb{J}}{\mathbb{J}_0}) + \mathbb{K} + \mathbb{L}. \tag{28}$$

From Lemma(3.3), we get

$$ {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \leq \left(1 - \frac{\mathbb{J}_0}{\mathbb{J}}\right) {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J} + {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K} + {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}. \tag{29}$$

Replacing ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}$, ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}$, ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}$ with their values from (4), we have

$$ {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \leq \left(1 - \frac{\mathbb{J}_0}{\mathbb{J}}\right)\left(\beta - \delta\frac{\mathbb{J}\mathbb{K}}{\mathbb{N}} + \eta\mathbb{L} - \phi\mathbb{J}\right) + \left(\delta\frac{\mathbb{J}\mathbb{K}}{\mathbb{N}} - (\lambda+\phi)\mathbb{K}\right) + \left(\lambda\mathbb{K} - (\eta+\phi)\mathbb{L}\right). \tag{30}$$

Assume $\mathbb{J} = \mathbb{J} - \mathbb{J}_0$, $\mathbb{K} = \mathbb{K} - \mathbb{K}_0$, $\mathbb{L} = \mathbb{L} - \mathbb{L}_0$ and after some calculations, we get

$$\begin{aligned} {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \leq\ & \beta\left(\frac{\mathbb{J}-\mathbb{J}_0}{\mathbb{J}}\right) - \delta\frac{(\mathbb{J}-\mathbb{J}_0)^2(\mathbb{K}-\mathbb{K}_0)}{\mathbb{J}\mathbb{N}} - \eta\left(\frac{\mathbb{J}_0}{\mathbb{J}}\right)(\mathbb{L}-\mathbb{L}_0) - \phi\frac{(\mathbb{J}-\mathbb{J}_0)^2}{\mathbb{J}} + \delta\frac{(\mathbb{J}-\mathbb{J}_0)(\mathbb{K}-\mathbb{K}_0)}{\mathbb{N}} \\ & - \phi(\mathbb{K}-\mathbb{K}_0) - \phi(\mathbb{L}-\mathbb{L}_0). \end{aligned} \tag{31}$$

While ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \leq 0$ for $\mathcal{R}_0 < 1$, and ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} = 0$ only when $\mathbb{J} = \mathbb{J}_0$, $\mathbb{K} = \mathbb{K}_0$, $\mathbb{L} = \mathbb{L}_0$. Hence, we can make conclusion that system (4) is globally asymptotically stable. □

Theorem 3.5. *If* $\mathcal{R}_0 > 1$*, then The endemic equilibrium states* $\mathcal{E}^\star$ *are globally asymptotically stable.*

Proof. Let

$$\mathtt{F} = \mathtt{Q}_1(\mathbb{J} - \mathbb{J}^\star - \mathbb{J}^\star \ln \frac{\mathbb{J}}{\mathbb{J}^\star}) + \mathtt{Q}_2(\mathbb{K} - \mathbb{K}^\star - \mathbb{K}^\star \ln \frac{\mathbb{K}}{\mathbb{K}^\star}) + \mathtt{Q}_3(\mathbb{L} - \mathbb{L}^\star - \mathbb{L}^\star \ln \frac{\mathbb{L}}{\mathbb{L}^\star}), \tag{32}$$

while $\mathtt{Q}_i$, $i = 1,2,3.$ are positive constants to be selected later. Then, we get

$$ {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \ \leq\ \mathtt{Q}_1\left(\frac{\mathbb{J}-\mathbb{J}^\star}{\mathbb{J}}\right) {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J} + \mathtt{Q}_2\left(\frac{\mathbb{K}-\mathbb{K}^\star}{\mathbb{K}}\right) {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K} + \mathtt{Q}_3\left(\frac{\mathbb{L}-\mathbb{L}^\star}{\mathbb{L}}\right) {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}. \tag{33}$$

Now substituting the values of ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}$, ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}$, ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}$ from (4) as given below;

$$\Big\{ {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \leq \mathtt{Q}_1\left(\tfrac{\mathbb{J}-\mathbb{J}^\star}{\mathbb{J}}\right)\left[\beta - \delta\tfrac{\mathbb{J}\mathbb{K}}{\mathbb{N}} + \eta\mathbb{L} - \phi\mathbb{J}\right] + \mathtt{Q}_2\left(\tfrac{\mathbb{K}-\mathbb{K}^\star}{\mathbb{K}}\right)\left[\delta\tfrac{\mathbb{J}\mathbb{K}}{\mathbb{N}} - (\lambda+\phi)\mathbb{K}\right] + \mathtt{Q}_3\left(\tfrac{\mathbb{L}-\mathbb{L}^\star}{\mathbb{L}}\right)\left[\lambda\mathbb{K} - (\eta+\phi)\mathbb{L}\right]. \tag{34}$$

using $\mathbb{J}=\mathbb{J}-\mathbb{J}^\star$, $\mathbb{K}=\mathbb{K}-\mathbb{K}^\star$, $\mathbb{L}=\mathbb{L}-\mathbb{L}^\star$, we have

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \leq & \mathtt{Q}_1\beta-\mathtt{Q}_1\beta\big(\frac{\mathbb{J}^\star}{\mathbb{J}}\big)-\mathtt{Q}_1\delta\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}\mathbb{N}}\mathbb{K}+\mathtt{Q}_1\delta\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}\mathbb{N}}\mathbb{K}^\star+\mathtt{Q}_1\eta\mathbb{L}-\mathtt{Q}_1\eta\big(\frac{\mathbb{J}^\star}{\mathbb{J}}\big)\mathbb{L}^\star-\mathtt{Q}_1\phi\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}} \\ & +\mathtt{Q}_2\delta\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}\mathbb{N}}\mathbb{J}-\mathtt{Q}_2\delta\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}\mathbb{N}}\mathbb{J}^\star-\mathtt{Q}_2(\lambda+\phi)\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}}+\mathtt{Q}_3\lambda\mathbb{K}-\mathtt{Q}_3\lambda\mathbb{K}^\star-\mathtt{Q}_3\lambda\big(\frac{\mathbb{L}^\star}{\mathbb{L}}\big)\mathbb{K} \\ & +\mathtt{Q}_3\lambda\big(\frac{\mathbb{L}^\star}{\mathbb{L}}\big)\mathbb{K}^\star-\mathtt{Q}_3(\eta+\phi)\frac{(\mathbb{L}-\mathbb{L}^\star)^2}{\mathbb{L}}. \end{cases} \tag{35}$$

Suppose $\mathtt{Q}_1=\mathtt{Q}_2=\mathtt{Q}_3=1$ then we have

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \leq \beta-\beta\big(\frac{\mathbb{J}^\star}{\mathbb{J}}\big)-\delta\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}\mathbb{N}}\mathbb{K}+\delta\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}\mathbb{N}}\mathbb{K}^\star+\eta\mathbb{L}-\eta\big(\frac{\mathbb{J}^\star}{\mathbb{J}}\big)\mathbb{L}^\star-\phi\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}}+\delta\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}\mathbb{N}}\mathbb{J} \\ \quad -\delta\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}\mathbb{N}}\mathbb{J}^\star-(\lambda+\phi)\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}}+\lambda\mathbb{K}-\lambda\mathbb{K}^\star-\lambda\big(\frac{\mathbb{L}^\star}{\mathbb{L}}\big)\mathbb{K}+\lambda\big(\frac{\mathbb{L}^\star}{\mathbb{L}}\big)\mathbb{K}^\star-(\eta+\phi)\frac{(\mathbb{L}-\mathbb{L}^\star)^2}{\mathbb{L}}. \end{cases} \tag{36}$$

For simplification, we can write

$${}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F} \leq \gamma_1-\gamma_2, \tag{37}$$

where

$$\begin{cases} \gamma_1=\beta+\delta\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}\mathbb{N}}\mathbb{K}^\star+\eta\mathbb{L}+\delta\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}\mathbb{N}}\mathbb{J}-\delta\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}\mathbb{N}}\mathbb{J}^\star+\lambda\mathbb{K}+\lambda\big(\frac{\mathbb{L}^\star}{\mathbb{L}}\big)\mathbb{K}^\star, \\ \gamma_2=-\beta\big(\frac{\mathbb{J}^\star}{\mathbb{J}}\big)-\delta\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}\mathbb{N}}\mathbb{K}-\eta\big(\frac{\mathbb{J}^\star}{\mathbb{J}}\big)\mathbb{L}^\star-\phi\frac{(\mathbb{J}-\mathbb{J}^\star)^2}{\mathbb{J}}-(\lambda+\phi)\frac{(\mathbb{K}-\mathbb{K}^\star)^2}{\mathbb{K}}-\lambda\mathbb{K}^\star-\lambda\big(\frac{\mathbb{L}^\star}{\mathbb{L}}\big)\mathbb{K} \\ \quad -(\eta+\phi)\frac{(\mathbb{L}-\mathbb{L}^\star)^2}{\mathbb{L}}. \end{cases} \tag{38}$$

We can see that if $\gamma_1<\gamma_2 \Longrightarrow {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}<0$. While, if we have $\mathbb{J}=\mathbb{J}^\star$,$\mathbb{K}=\mathbb{K}^\star$,$\mathbb{L}=\mathbb{L}^\star$ then we examine $\gamma_1-\gamma_2=0 \Longrightarrow {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}=0$. We investigate that the greatest compact invariant set in the proposed model in

$$\Big\{(\mathbb{J}^\star,\mathbb{K}^\star,\mathbb{L}^\star)\in\mathbf{W}: {}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}=0\Big\} \tag{39}$$

is the point $\mathcal{E}^\star$, the endemic equilibrium. Therefore, we con make conclusion that $\mathsf{E}^\star$ is globally asymptotically stable in $\mathbf{W}$ if $\gamma_1<\gamma_2$. □

3.6. Second Derivative of Lyapunov

More observation on the details of each variation is required because the first derivative evaluation of an arbitrary function cannot fully illustrate its variations. As a result, we investigate the system's associated Lyapunov function's second derivative as follows:

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}\big) \leq {}_0^{CPC}\mathsf{D}_t^{\mu}\Big[\mathtt{Q}_1\big(\frac{\mathbb{J}-\mathbb{J}^\star}{\mathbb{J}}\big){}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}+\mathtt{Q}_2\big(\frac{\mathbb{K}-\mathbb{K}^\star}{\mathbb{K}}\big){}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}+\mathtt{Q}_3\big(\frac{\mathbb{L}-\mathbb{L}^\star}{\mathbb{L}}\big){}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}\Big] \\ \quad \leq \mathtt{Q}_1\Big(\frac{{}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}}{\mathbb{J}}\Big)^2\mathbb{J}^\star+\mathtt{Q}_2\Big(\frac{{}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}}{\mathbb{K}}\Big)^2\mathbb{K}^\star+\mathtt{Q}_3\Big(\frac{{}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}}{\mathbb{L}}\Big)^2\mathbb{L}^\star+\mathtt{Q}_1\big(1-\frac{\mathbb{J}^\star}{\mathbb{J}}\big){}_0^{CPC}\mathsf{D}_t^{\mu}\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathsf{J}\big) \\ \quad +\mathtt{Q}_2\big(1-\frac{\mathbb{K}^\star}{\mathbb{K}}\big){}_0^{CPC}\mathsf{D}_t^{\mu}\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathsf{K}\big)+\mathtt{Q}_3\big(1-\frac{\mathbb{L}^\star}{\mathbb{L}}\big){}_0^{CPC}\mathsf{D}_t^{\mu}\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathsf{L}\big), \end{cases} \tag{40}$$

where

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}(t)\big)=\beta-\delta\frac{\big(({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J})\mathbb{K}+\mathbb{J}({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K})\big)\mathbb{N}-({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{N})\mathbb{J}\mathbb{K}}{\mathbb{N}^2}+\eta\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}\big)-\phi\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}\big), \\ {}_0^{CPC}\mathsf{D}_t^{\mu}\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}(t)\big)=\delta\frac{\big(({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J})\mathbb{K}+\mathbb{J}({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K})\big)\mathbb{N}-({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{N})\mathbb{J}\mathbb{K}}{\mathbb{N}^2}-(\lambda+\phi)\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}\big), \\ {}_0^{CPC}\mathsf{D}_t^{\mu}\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}(t)\big)=\lambda\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}\big)-(\eta+\phi)\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}\big). \end{cases} \tag{41}$$

Then we obtain

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\big[{}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}\big] \leq \dot{\Theta}(\mathbb{J},\mathbb{K},\mathbb{L}) \\ \quad +\mathtt{Q}_1\big(1-\frac{\mathbb{J}^\star}{\mathbb{J}}\big)\Big[\beta-\delta\frac{\big(({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J})\mathbb{K}+\mathbb{J}({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K})\big)\mathbb{N}-({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{N})\mathbb{J}\mathbb{K}}{\mathbb{N}^2}+\eta\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}\big)-\phi\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}\big)\Big], \\ \quad +\mathtt{Q}_2\big(1-\frac{\mathbb{K}^\star}{\mathbb{K}}\big)\Big[\delta\frac{\big(({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J})\mathbb{K}+\mathbb{J}({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K})\big)\mathbb{N}-({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{N})\mathbb{J}\mathbb{K}}{\mathbb{N}^2}-(\lambda+\phi)\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}\big)\Big], \\ \quad +\mathtt{Q}_3\big(1-\frac{\mathbb{L}^\star}{\mathbb{L}}\big)\Big[\lambda\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}\big)-(\eta+\phi)\big({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}\big)\Big], \end{cases} \tag{42}$$

Where

$$\dot{\Theta}(\mathbb{J},\mathbb{K},\mathbb{L},\mathcal{Y},\mathcal{Z},\mathbb{R}) = \mathtt{Q}_1\Big(\frac{{}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}}{\mathbb{J}}\Big)^2\mathbb{J}^{\star} + \mathtt{Q}_2\Big(\frac{{}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}}{\mathbb{K}}\Big)^2\mathbb{K}^{\star} + \mathtt{Q}_3\Big(\frac{{}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}}{\mathbb{L}}\Big)^2\mathbb{L}^{\star}. \tag{43}$$

now replace ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}$, ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}$, ${}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}$ in equation with their values from the system (4). After simplifying and differentiating resulting equation into sum of positive expressions denoted by φ_1 and negative expressions denoted by φ_2, we can express

$${}_0^{CPC}\mathsf{D}_t^{\mu}[\,{}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}] \leq \varphi_1 - \varphi_2. \tag{44}$$

We examine that
${}_0^{CPC}\mathsf{D}_t^{\mu}[\,{}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}] > 0$ if $\varphi_1 > \varphi_2$, ${}_0^{CPC}\mathsf{D}_t^{\mu}[\,{}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}] < 0$ if $\varphi_1 < \varphi_2$, and ${}_0^{CPC}\mathsf{D}_t^{\mu}[\,{}_0^{CPC}\mathsf{D}_t^{\mu}\mathtt{F}] = 0$ if $\varphi_1 = \varphi_2$.

3.7. Analysis of Proposed Model

The inverse operators for the PC and CPC operators are given by

$$\begin{cases} {}_0^{PC}\mathsf{I}_t^{\mu}\mathbb{J}(t) = \int_0^t \exp(-\int_{\varepsilon}^t \frac{\mathcal{A}_1(\mu,v)}{\mathcal{A}_0(\mu,v)}dv)\frac{{}_0^{RL}\mathsf{D}_{\mu}^{1-\mu}\mathbb{J}(\varepsilon)}{\mathcal{A}_0(\mu,\varepsilon)}d\varepsilon\ ,\ {}_0^{CPC}\mathsf{I}_t^{\mu}\mathbb{J}(t) = \frac{1}{\mathscr{F}_0(\mu)}\int_0^t \exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}(t-\varepsilon)){}_0^{RL}\mathsf{D}_{\varepsilon}^{1-\mu}\mathbb{J}(\varepsilon)d\varepsilon, \\ {}_0^{PC}\mathsf{I}_t^{\mu}\mathbb{K}(t) = \int_0^t \exp(-\int_{\varepsilon}^t \frac{\mathcal{A}_1(\mu,v)}{\mathcal{A}_0(\mu,v)}dv)\frac{{}_0^{RL}\mathsf{D}_{\mu}^{1-\mu}\mathbb{K}(\varepsilon)}{\mathcal{A}_0(\mu,\varepsilon)}d\varepsilon\ ,\ {}_0^{CPC}\mathsf{I}_t^{\mu}\mathbb{K}(t) = \frac{1}{\mathscr{F}_0(\mu)}\int_0^t \exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}(t-\varepsilon)){}_0^{RL}\mathsf{D}_{\varepsilon}^{1-\mu}\mathbb{K}(\varepsilon)d\varepsilon, \\ {}_0^{PC}\mathsf{I}_t^{\mu}\mathbb{L}(t) = \int_0^t \exp(-\int_{\varepsilon}^t \frac{\mathcal{A}_1(\mu,v)}{\mathcal{A}_0(\mu,v)}dv)\frac{{}_0^{RL}\mathsf{D}_{\mu}^{1-\mu}\mathbb{L}(\varepsilon)}{\mathcal{A}_0(\mu,\varepsilon)}d\varepsilon\ ,\ {}_0^{CPC}\mathsf{I}_t^{\mu}\mathbb{L}(t) = \frac{1}{\mathscr{F}_0(\mu)}\int_0^t \exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}(t-\varepsilon)){}_0^{RL}\mathsf{D}_{\varepsilon}^{1-\mu}\mathbb{L}(\varepsilon)d\varepsilon, \end{cases} \tag{45}$$

that ensure the following inversion relations [28]:

$$\begin{cases} {}_0^{PC}\mathsf{D}_t^{\mu}({}_0^{PC}\mathsf{I}_t^{\mu}\mathbb{J}(t)) = {}_0^{CPC}\mathsf{D}_t^{\mu}({}_0^{CPC}\mathsf{I}_t^{\mu}\mathbb{J}(t)) = \mathbb{J}(t) - \frac{t^{-\mu}}{\Gamma(1-\mu)}\lim_{t\to 0}{}_0^{RL}\mathsf{D}_t^{\mu}\mathbb{J}(t), \\ {}_0^{PC}\mathsf{D}_t^{\mu}({}_0^{PC}\mathsf{I}_t^{\mu}\mathbb{K}(t)) = {}_0^{CPC}\mathsf{D}_t^{\mu}({}_0^{CPC}\mathsf{I}_t^{\mu}\mathbb{K}(t)) = \mathbb{K}(t) - \frac{t^{-\mu}}{\Gamma(1-\mu)}\lim_{t\to 0}{}_0^{RL}\mathsf{D}_t^{\mu}\mathbb{K}(t), \\ {}_0^{PC}\mathsf{D}_t^{\mu}({}_0^{PC}\mathsf{I}_t^{\mu}\mathbb{L}(t)) = {}_0^{CPC}\mathsf{D}_t^{\mu}({}_0^{CPC}\mathsf{I}_t^{\mu}\mathbb{L}(t)) = \mathbb{L}(t) - \frac{t^{-\mu}}{\Gamma(1-\mu)}\lim_{t\to 0}{}_0^{RL}\mathsf{D}_t^{\mu}\mathbb{L}(t), \end{cases} \tag{46}$$

$$\begin{cases} {}_0^{PC}\mathsf{I}_t^{\mu}({}_0^{PC}\mathsf{D}_t^{\mu}\mathbb{J}(t)) = \mathbb{J}(t) - \exp(-\int_0^t \frac{\mathcal{A}_1(\mu,v)}{\mathcal{A}_0(\mu,v)}dv)\mathbb{J}(0)\ ,\ {}_0^{CPC}\mathsf{I}_t^{\mu}({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}(t)) = \mathbb{J}(t) - \exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t)\mathbb{J}(0), \\ {}_0^{PC}\mathsf{I}_t^{\mu}({}_0^{PC}\mathsf{D}_t^{\mu}\mathbb{K}(t)) = \mathbb{K}(t) - \exp(-\int_0^t \frac{\mathcal{A}_1(\mu,v)}{\mathcal{A}_0(\mu,v)}dv)\mathbb{K}(0)\ ,\ {}_0^{CPC}\mathsf{I}_t^{\mu}({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}(t)) = \mathbb{K}(t) - \exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t)\mathbb{K}(0), \\ {}_0^{PC}\mathsf{I}_t^{\mu}({}_0^{PC}\mathsf{D}_t^{\mu}\mathbb{L}(t)) = \mathbb{L}(t) - \exp(-\int_0^t \frac{\mathcal{A}_1(\mu,v)}{\mathcal{A}_0(\mu,v)}dv)\mathbb{L}(0)\ ,\ {}_0^{CPC}\mathsf{I}_t^{\mu}({}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}(t)) = \mathbb{L}(t) - \exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t)\mathbb{L}(0). \end{cases} \tag{47}$$

Proof. We can express (45) as operational composition as follows;

$${}_0^{PC}\mathsf{I}_t^{\mu} = {}^{P}\mathsf{I}_t^{\mu} \bullet {}_0^{RL}\mathsf{D}_t^{1-\mu} \quad , \quad {}_0^{CPC}\mathsf{I}_t^{\mu} = {}^{CP}\mathsf{I}_t^{\mu} \bullet {}_0^{RL}\mathsf{D}_t^{1-\mu}. \tag{48}$$

Hence, we have for class $\mathbb{J}(t)$:

$$\begin{cases} ({}_0^{PC}\mathsf{D}_t^{\mu} \bullet {}_0^{PC}\mathsf{I}_t^{\mu})\mathbb{J}(t) = ({}_0^{RL}\mathsf{I}_t^{1-\mu} \bullet {}^{P}\mathsf{D}_t^{\mu}) \bullet ({}^{P}\mathsf{I}_t^{\mu} \bullet {}_0^{RL}\mathsf{D}_t^{1-\mu})\mathbb{J}(t) = ({}_0^{RL}\mathsf{I}_t^{1-\mu} \bullet {}_0^{RL}\mathsf{D}_t^{1-\mu})\mathbb{J}(t) \\ \qquad = \mathbb{J}(t) - \frac{t^{-\mu}}{\Gamma(1-\mu)}\lim_{t\to 0}{}_0^{RL}\mathsf{D}_t^{\mu}\mathbb{J}(t), \\ ({}_0^{PC}\mathsf{I}_t^{\mu} \bullet {}_0^{PC}D_t^{\mu})\mathbb{J}(t) = ({}^{P}I_t^{\mu} \bullet {}_0^{RL}D_t^{1-\mu}) \bullet ({}_0^{RL}I_t^{1-\mu} \bullet {}^{P}D_t^{1-\mu})\mathbb{J}(t) = ({}^{S}\mathsf{I}_t^{\mu} \bullet {}^{P}\mathsf{D}_t^{\mu})\mathbb{J}(t) \\ \qquad = \mathbb{J}(t) - \exp(-\int_0^t \frac{\mathcal{A}_1(\mu,v)}{\mathcal{A}_0(\mu,v)}dv)\mathbb{J}(0), \\ ({}_0^{CPC}\mathsf{D}_t^{\mu} \bullet {}_0^{CPC}\mathsf{I}_t^{\mu})\mathbb{J}(t) = ({}_0^{RL}\mathsf{I}_t^{1-\mu} \bullet {}^{CP}\mathsf{D}_t^{\mu}) \bullet ({}^{CP}\mathsf{I}_t^{\mu} \bullet {}_0^{RL}\mathsf{D}_t^{1-\mu})\mathbb{J}(t) = ({}_0^{RL}\mathsf{I}_t^{1-\mu} \bullet {}_0^{RL}\mathsf{D}_t^{1-\mu})\mathbb{J}(t) \\ \qquad = \mathbb{P}(t) - \frac{t^{-\mu}}{\Gamma(1-\mu)}\lim_{t\to 0}{}_0^{RL}\mathsf{D}_t^{\mu}\mathbb{J}(t), \\ ({}_0^{CPC}\mathsf{I}_t^{\mu} \bullet {}_0^{CPC}\mathsf{D}_t^{\mu})\mathbb{J}(t) = ({}^{CP}\mathsf{I}_t^{\mu} \bullet {}_0^{RL}\mathsf{D}_t^{1-\mu}) \bullet ({}_0^{RL}\mathsf{I}_t^{1-\mu} \bullet {}^{CP}\mathsf{D}_t^{1-\mu})\mathbb{J}(t) = ({}^{CP}\mathsf{I}_t^{\mu} \bullet {}^{CP}\mathsf{D}_t^{\mu})\mathbb{J}(t) \\ \qquad = \mathbb{J}(t) - \exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t)\mathbb{J}(0). \end{cases}$$

This method allows us to demonstrate it for other classes. □

Lemma 3.6. *For $\Pi = m+n-ij$, we can simplify the operator $\mathsf{D}_a^{m,n,\Pi}$ as given below:*

$$\begin{cases} \mathsf{D}_a^{m,n,\mu}\mathbb{J}(t) = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\mu}\ \mathsf{I}_a^{(1-\Pi),\mu}\mathbb{J}(t) = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\Pi,\mu}\mathbb{J}(t), \\ \mathsf{D}_a^{m,n,\mu}\mathbb{K}(t) = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\mu}\ \mathsf{I}_a^{(1-\Pi),\mu}\mathbb{K}(t) = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\Pi,\mu}\mathbb{K}(t),. \\ \mathsf{D}_a^{m,n,\mu}\mathbb{L}(t) = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\mu}\ \mathsf{I}_a^{(1-\Pi),\mu}\mathbb{L}(t) = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\Pi,\mu}\mathbb{L}(t) \end{cases} \tag{49}$$

Proof. Utilizing definitions [37], we find

$$\begin{cases} \mathsf{D}_a^{m,n,\mu}\mathbb{J}(t) = \mathsf{I}_a^{n(1-m),\mu}\left[\mathsf{D}_a^{\mu}(\mathsf{I}_a^{(1-n)(1-m),\mu})\right]\mathbb{J}(t), \\ \mathsf{D}_a^{m,n,\mu}\mathbb{K}(t) = \mathsf{I}_a^{n(1-m),\mu}\left[\mathsf{D}_a^{\mu}(\mathsf{I}_a^{(1-n)(1-m),\mu})\right]\mathbb{K}(t), \\ \mathsf{D}_a^{m,n,\mu}\mathbb{L}(t) = \mathsf{I}_a^{n(1-m),\mu}\left[\mathsf{D}_a^{\mu}(\mathsf{I}_a^{(1-n)(1-m),\mu})\right]\mathbb{L}(t), \end{cases}$$

$$\begin{cases} = \mathsf{I}_a^{n(1-m),\mu}\left[\frac{\mathsf{D}_a^{\mu}}{\mu^{(1-\Pi)}\Gamma(1-\Pi)}\int_a^t e^{\frac{\mu-1}{\mu}(t-\varepsilon)}(t-\varepsilon)^{(1-\Pi)-1}\mathbb{J}(\varepsilon)d\varepsilon\right] = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\Pi,\mu}\mathbb{J}(t), \\ = \mathsf{I}_a^{n(1-m),\mu}\left[\frac{\mathsf{D}_a^{\mu}}{\mu^{(1-\Pi)}\Gamma(1-\Pi)}\int_a^t e^{\frac{\mu-1}{\mu}(t-\varepsilon)}(t-\varepsilon)^{(1-\Pi)-1}\mathbb{K}(\varepsilon)d\varepsilon\right] = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\Pi,\mu}\mathbb{K}(t), \\ = \mathsf{I}_a^{n(1-m),\mu}\left[\frac{\mathsf{D}_a^{\mu}}{\mu^{(1-\Pi)}\Gamma(1-\Pi)}\int_a^t e^{\frac{\mu-1}{\mu}(t-\varepsilon)}(t-\varepsilon)^{(1-\Pi)-1}\mathbb{L}(\varepsilon)d\varepsilon\right] = \mathsf{I}_a^{n(1-m),\mu}\ \mathsf{D}_a^{\Pi,\mu}\mathbb{L}(t). \end{cases} \tag{50}$$

□

Lemma 3.7. *Consider $0 < m < 1$, $\mu \in (0,1]$, $0 \le n \le 1$, and $\Pi = m+n-mn$. If $\{\mathbb{J},\mathbb{K},\mathbb{L}\} \in \mathbf{C}_{1-\Pi}^{\Pi}[a,b]$ then*

$$\begin{cases} \mathsf{I}_t^{\Pi,\mu}\ \mathsf{D}_t^{\Pi,\mu}\mathbb{J}(t) = \mathsf{I}_t^{m,\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{J}(t) \quad , \quad \mathsf{D}_t^{\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{J}(t) = \mathsf{D}_t^{n(1-m),\mu}\mathbb{J}(t), \\ \mathsf{I}_t^{\Pi,\mu}\ \mathsf{D}_t^{\Pi,\mu}\mathbb{K}(t) = \mathsf{I}_t^{m,\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{K}(t) \quad , \quad \mathsf{D}_t^{\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{K}(t) = \mathsf{D}_t^{n(1-m),\mu}\mathbb{K}(t), \\ \mathsf{I}_t^{\Pi,\mu}\ \mathsf{D}_t^{\Pi,\mu}\mathbb{L}(t) = \mathsf{I}_t^{m,\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{L}(t) \quad , \quad \mathsf{D}_t^{\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{L}(t) = \mathsf{D}_t^{n(1-m),\mu}\mathbb{L}(t). \end{cases} \tag{51}$$

Proof. utilizing Lemma (3.6)

$$\begin{cases} \mathsf{I}_t^{\Pi,\mu}\ \mathsf{D}_t^{\Pi,\mu}\mathbb{J}(t) = \mathsf{I}_t^{\Pi,\mu}\left[\mathsf{I}_t^{-n(1-m),\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{J}(t)\right] = \mathsf{I}_t^{m+n-mn,\mu}\ \mathsf{I}_t^{-n(1-m),\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{J}(t) = \mathsf{I}_t^{m,\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{J}(t), \\ \mathsf{I}_t^{\Pi,\mu}\ \mathsf{D}_t^{\Pi,\mu}\mathbb{K}(t) = \mathsf{I}_t^{\Pi,\mu}\left[\mathsf{I}_t^{-n(1-m),\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{K}(t)\right] = \mathsf{I}_t^{m+n-mn,\mu}\ \mathsf{I}_t^{-n(1-m),\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{K}(t) = \mathsf{I}_t^{m,\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{K}(t), \\ \mathsf{I}_t^{\Pi,\mu}\ \mathsf{D}_t^{\Pi,\mu}\mathbb{L}(t) = \mathsf{I}_t^{\Pi,\mu}\left[\mathsf{I}_t^{-n(1-m),\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{L}(t)\right] = \mathsf{I}_t^{m+n-mn,\mu}\ \mathsf{I}_t^{-n(1-m),\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{L}(t) = \mathsf{I}_t^{m,\mu}\ \mathsf{D}_t^{m,n,\mu}\mathbb{L}(t). \end{cases} \tag{52}$$

Additionally, by definition[30], we find that

$$\begin{cases} \mathsf{D}_t^{\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{J}(t) = \mathsf{D}_t^{\mu}\ \mathsf{I}_t^{1-\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{J}(t) = \mathsf{D}_t^{\mu}\ \mathsf{I}_t^{m-n+mn,\mu}\mathbb{J}(t) = \mathsf{D}_t^{n(1-m),\mu}\mathbb{J}(t), \\ \mathsf{D}_t^{\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{K}(t) = \mathsf{D}_t^{\mu}\ \mathsf{I}_t^{1-\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{K}(t) = \mathsf{D}_t^{\mu}\ \mathsf{I}_t^{m-n+mn,\mu}\mathbb{K}(t) = \mathsf{D}_t^{n(1-m),\mu}\mathbb{K}(t), \\ \mathsf{D}_t^{\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{L}(t) = \mathsf{D}_t^{\mu}\ \mathsf{I}_t^{1-\Pi,\mu}\ \mathsf{I}_t^{m,\mu}\mathbb{L}(t) = \mathsf{D}_t^{\mu}\ \mathsf{I}_t^{m-n+mn,\mu}\mathbb{L}(t) = \mathsf{D}_t^{n(1-m),\mu}\mathbb{L}(t). \end{cases} \tag{53}$$

□

3.8. *Solution of system (4) with CPC operator*

Let

$$\begin{cases} {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{J}(t) = \beta - \xi\mathbb{J}(t)\mathbb{K}(t) + \eta\mathbb{L}(t) - \phi\mathbb{J}(t), \\ {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{K}(t) = \xi\mathbb{J}(t)\mathbb{K}(t) - (\lambda+\phi)\mathbb{K}(t), \\ {}_0^{CPC}\mathsf{D}_t^{\mu}\mathbb{L}(t) = \lambda\mathbb{K}(t) - (\eta+\phi)\mathbb{L}(t), \end{cases} \tag{54}$$

with non-negative initial constraints,

$$\mathbb{J}(0) = \mathbb{J}^0 \ , \ \mathbb{K}(0) = \mathbb{K}^0 \ , \ \mathbb{L}(0) = \mathbb{L}^0, \tag{55}$$

where $\xi = \frac{\delta}{\mathbb{N}}$. Applying Laplace Transform, we have

$$\begin{cases} \mathscr{L}[\mathbb{J}(t)] = \frac{1}{\mathcal{A}_1(\mu)s^{\mu-1}+\mathcal{A}_0(\mu)s^{\mu}}\Big(\mathcal{A}_0(\mu)s^{\mu-1}\mathbb{J}_0 + \frac{\beta}{s} - \xi\mathscr{L}[\mathbb{J}(t)\mathbb{K}(t)] + \eta\mathscr{L}[\mathbb{L}(t)] - \phi\mathscr{L}[\mathbb{J}(t)]\Big), \\ \mathscr{L}[\mathbb{K}(t)] = \frac{1}{\mathcal{A}_1(\mu)s^{\mu-1}+\mathcal{A}_0(\mu)s^{\mu}}\Big(\mathcal{A}_0(\mu)s^{\mu-1}\mathbb{K}_0 + \xi\mathscr{L}[\mathbb{J}(t)\mathbb{K}(t)] - (\lambda+\phi)\mathscr{L}[\mathbb{K}(t)]\Big), \\ \mathscr{L}[\mathbb{L}(t)] = \frac{1}{\mathcal{A}_1(\mu)s^{\mu-1}+\mathcal{A}_0(\mu)s^{\mu}}\Big(\mathcal{A}_0(\mu)s^{\mu-1}\mathbb{L}_0 + \lambda\mathscr{L}[\mathbb{K}(t)] - (\eta+\phi)\mathscr{L}[\mathbb{L}(t)]\Big). \end{cases} \tag{56}$$

Equivalently

$$\begin{cases} \mathscr{L}[\mathbb{J}(t)] = \frac{\mathbb{J}^0}{s+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}} + \sum_{n=0}^{\infty}\frac{(-\mathcal{A}_1(\mu))^n}{(\mathscr{F}_0(\mu))^{n+1}}s^{-\mu-n}\Big(\frac{\beta}{s} - \xi\mathscr{L}[\mathbb{J}(t)\mathbb{K}(t)] + \eta\mathscr{L}[\mathbb{L}(t)] - \phi\mathscr{L}[\mathbb{J}(t)]\Big), \\ \mathscr{L}[\mathbb{K}(t)] = \frac{\mathbb{K}^0}{s+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}} + \sum_{n=0}^{\infty}\frac{(-\mathcal{A}_1(\mu))^n}{(\mathscr{F}_0(\mu))^{n+1}}s^{-\mu-n}\Big(\xi\mathscr{L}[\mathbb{J}(t)\mathbb{K}(t)] - (\lambda+\phi)\mathscr{L}[\mathbb{K}](t)\Big), \\ \mathscr{L}[\mathbb{L}(t)] = \frac{\mathbb{L}^0}{s+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}} + \sum_{n=0}^{\infty}\frac{(-\mathcal{A}_1(\mu))^n}{(\mathscr{F}_0(\mu))^{n+1}}s^{-\mu-n}\Big(\lambda\mathscr{L}[\mathbb{K}(t)] - (\eta+\phi)\mathscr{L}[\mathbb{L}(t)]\Big). \end{cases} \tag{57}$$

Consider that the method gives the outcomes as an infinite series;

$$\mathbb{J}(t) = \sum_{k=0}^{\infty}\mathbb{J}_k \quad , \quad \mathbb{K}(t) = \sum_{k=0}^{\infty}\mathbb{K}_k \quad , \quad \mathbb{L}(t) = \sum_{k=0}^{\infty}\mathbb{L}_k. \tag{58}$$

$$\mathbb{J}(t)\tilde{I}(t) = \sum_{k=0}^{\infty}\mathrm{Z}_k \quad , \quad \mathrm{Z}_k = \frac{1}{k!}\Big(\frac{d}{d\Lambda}\Big)^k\Big[\sum_{j=0}^{k}\Lambda^j\mathbb{J}_j\ \sum_{j=0}^{k}\Lambda^j\tilde{I}_j\Big]_{\Lambda=0}, \quad k = 0,1,2,3,\cdots. \tag{59}$$

using equation (58) and (59), and applying Inverse Laplace on both sides of equation (57) , we obtain

$$\begin{cases} \mathbb{J}_0(t) = \mathbb{J}^0\exp(-\frac{\mathcal{A}_1(\mu)}{\mathscr{F}_0(\mu)}t) + \frac{1}{\mathscr{F}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathscr{F}_0(\mu)}\big)^n\frac{\beta t^{\mu+n}}{\Gamma(\mu+n+1)}, \\ \mathbb{K}_0(t) = \mathbb{K}^0\exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t) \quad , \quad \mathbb{L}_0(t) = \mathbb{L}^0\exp(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t), \end{cases} \tag{60}$$

and for $k \geq 0$,

$$\begin{cases} \mathbb{J}_{k+1}(t) = \frac{1}{\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\frac{t^{\mu+n-1}}{\Gamma(\mu+n)}\mathscr{L}^{-1}\Big(-\xi\mathscr{L}[\mathrm{Z}_k] + \eta\mathscr{L}[\mathbb{L}_k] - \phi\mathscr{L}[\mathbb{J}_k]\Big), \\ \mathbb{K}_{k+1}(t) = \frac{1}{\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\frac{t^{\mu+n-1}}{\Gamma(\mu+n)}\mathscr{L}^{-1}\Big(\xi\mathscr{L}[\mathrm{Z}_k] - (\lambda+\phi)\mathscr{L}[\mathbb{K}_k]\Big), \\ \mathbb{L}_{k+1}(t) = \frac{1}{\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\frac{t^{\mu+n-1}}{\Gamma(\mu+n)}\mathscr{L}^{-1}\Big(\lambda\mathscr{L}[\mathbb{K}_k] - (\eta+\phi)\mathscr{L}[\mathbb{L}_k]\Big). \end{cases} \tag{61}$$

4. Transmission Dynamics of Skin Sores model with CPABC operator

Let

$$\begin{cases} {}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{J}(t) = \beta - \xi\mathcal{J}(t)\mathcal{K}(t) + \eta\mathcal{L}(t) - \phi\mathcal{J}(t), \\ {}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{K}(t) = \xi\mathcal{J}(t)\mathcal{K}(t) - (\lambda+\phi)\mathcal{K}(t), \\ {}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{L}(t) = \lambda\mathcal{K}(t) - (\eta+\phi)\mathcal{L}(t). \end{cases} \tag{62}$$

with

$$\mathcal{J}(0) = \mathcal{J}^0 \ , \ \mathcal{K}(0) = \mathcal{K}^0 \ , \ \mathcal{L}(0) = \mathcal{L}^0. \tag{63}$$

Theorem 4.1. *The Laplace Transform of CPABC operator is defined [29] as:*

$$\begin{cases}\mathscr{L}[{}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{J}(t)] = (\frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)s^{\mu-1}+s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_1(\mu)}{\mu+s^{\mu}(1-\mu)})\mathscr{L}[\mathcal{J}(t)] - (\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathcal{J}^0,\\ \mathscr{L}[{}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{K}(t)] = (\frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)s^{\mu-1}+s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_1(\mu)}{\mu+s^{\mu}(1-\mu)})\mathscr{L}[\mathcal{K}(t)] - (\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathcal{K}^0,\\ \mathscr{L}[{}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{L}(t)] = (\frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)s^{\mu-1}+s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_1(\mu)}{\mu+s^{\mu}(1-\mu)})\mathscr{L}[\mathcal{L}(t)] - (\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathcal{L}^0.\end{cases} \tag{64}$$

Proof. According to equation (2), we have

$$\begin{cases}\mathscr{L}[{}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{J}(t)] = \frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[\mathcal{J}(t)]\cdot\frac{s^{\mu-1}(1-\mu)}{\mu+s^{\mu}(1-\mu)} + \frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}{1-\mu}(s\mathscr{L}[\mathcal{J}(t)]-\mathcal{J}^0)\cdot\frac{s^{\mu-1}(1-\mu)}{\mu+s^{\mu}(1-\mu)},\\ \mathscr{L}[{}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{K}(t)] = \frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[\mathcal{K}(t)]\cdot\frac{s^{\mu-1}(1-\mu)}{\mu+s^{\mu}(1-\mu)} + \frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}{1-\mu}(s\mathscr{L}[\mathcal{K}(t)]-\mathcal{K}^0)\cdot\frac{s^{\mu-1}(1-\mu)}{\mu+s^{\mu}(1-\mu)},\\ \mathscr{L}[{}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{L}(t)] = \frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[\mathcal{L}(t)]\cdot\frac{s^{\mu-1}(1-\mu)}{\mu+s^{\mu}(1-\mu)} + \frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}{1-\mu}(s\mathscr{L}[\mathcal{L}(t)]-\mathcal{L}^0)\cdot\frac{s^{\mu-1}(1-\mu)}{\mu+s^{\mu}(1-\mu)}.\end{cases}$$

$$\begin{cases}= (\frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathscr{L}[\mathcal{J}(t)] + (\frac{s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)})\mathscr{L}[\mathcal{J}(t)] - (\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathcal{J}^0,\\ = (\frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathscr{L}[\mathcal{K}(t)] + (\frac{s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)})\mathscr{L}[\mathcal{K}(t)] - (\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathcal{K}^0,\\ = (\frac{\mathbb{AB}(\mu)\mathcal{A}_1(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathscr{L}[\mathcal{L}(t)] + (\frac{s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)})\mathscr{L}[\mathcal{L}(t)] - (\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)})\mathcal{L}^0.\end{cases}$$

□

Theorem 4.2. *[29] Let*

$$\begin{cases}{}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{J}(t) = \mathrm{C}_1(t),\\ {}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{K}(t) = \mathrm{C}_2(t),\\ {}_0^{CPABC}\mathrm{D}_t^{\mu}\mathcal{L}(t) = \mathrm{C}_3(t).\end{cases} \tag{65}$$

Applying Laplace Transform to the both sides of equations and assuming $\mathcal{J}(0) = \mathcal{K}(0) = \mathcal{L}(0) = 0$,

$$\begin{cases}\mathcal{J}(t) = \frac{\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)})^n {}_0\mathrm{I}_t^{\mu+n}\mathrm{C}_1(t) + \frac{1-\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)})^n {}_0\mathrm{I}_t^{n}\mathrm{C}_1(t),\\ \mathcal{K}(t) = \frac{\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)})^n {}_0\mathrm{I}_t^{\mu+n}\mathrm{C}_2(t) + \frac{1-\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)})^n {}_0\mathrm{I}_t^{n}\mathrm{C}_2(t),\\ \mathcal{L}(t) = \frac{\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)})^n {}_0\mathrm{I}_t^{\mu+n}\mathrm{C}_3(t) + \frac{1-\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)})^n {}_0\mathrm{I}_t^{n}\mathrm{C}_3(t).\end{cases} \tag{66}$$

Proof. Consider first Equation from (65) and utilizing Theorem (4.1), we have

$$\begin{aligned}\mathscr{L}[\mathcal{J}(t)] &= \frac{\mu+s^{\mu}(1-\mu)}{\mathbb{AB}(\mu)s^{\mu-1}[\mathcal{A}_1(\mu)+s\mathcal{A}_0(\mu)]}\mathscr{L}[\mathrm{C}_1(t)]\\ &= \frac{\mu+s^{\mu}(1-\mu)}{s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_0(\mu)[1+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}s^{-1}]}\mathscr{L}[\mathrm{C}_1(t)].\end{aligned} \tag{67}$$

Equivalently

$$\mathscr{L}[\mathcal{J}(t)] = [\frac{\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)})^n s^{-n-\mu} + \frac{1-\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)})^n s^{-n}]\mathscr{L}[\mathrm{C}_1(t)]. \tag{68}$$

We have, for $i = 1,2,3$,

$$\begin{cases}s^{-n-1}\mathscr{L}[\mathrm{C}_i(t)] = \mathscr{L}(\frac{t^{\mu+n-1}}{\Gamma(\mu+n)})\mathscr{L}[\mathrm{C}_i(t) = \mathscr{L}\big(\mathrm{C}_i(t)\circ\frac{t^{\mu+n-1}}{\Gamma(\mu+n)}\big) = \mathscr{L}[{}_0\mathrm{I}_t^{n+\mu}\mathrm{C}_i(t)]\\ s^{-n}\mathscr{L}[\mathrm{C}_i(t)] = \mathscr{L}(\frac{t^n}{\Gamma(n)})\mathscr{L}[\mathrm{C}_i(t) = \mathscr{L}\big(\mathrm{C}_i(t)\circ\frac{t^{n-1}}{\Gamma(n)}\big) = \mathscr{L}[{}_0\mathrm{I}_t^{n}\mathrm{C}_i(t)].\end{cases} \tag{69}$$

Hence

$$\mathscr{L}[\mathcal{J}(t)] = \frac{\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}[{}_0\mathrm{I}_t^{\mu+n}\mathrm{C}_1(t)] + \frac{1-\mu}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}[{}_0\mathrm{I}_t^{n}\mathrm{C}_1(t)]. \tag{70}$$

Applying Inverse Laplace transform, we have the required result. □

Using Laplace Transform on both sides of equations and utilizing Theorem (4.1), we get:

$$\begin{cases}\mathscr{L}\big[\mathcal{J}(t)\big] = \frac{\mu+s^{\mu}(1-\mu)}{s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_0(\mu)\big(1+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}s^{-1}\big)}\Big(\big(\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)}\big)\mathcal{J}^0 + \mathscr{L}\Big[\beta-\xi\mathcal{J}(t)\mathcal{K}(t)+\eta\mathcal{L}(t)-\phi\mathcal{J}(t)\Big]\Big),\\ \mathscr{L}\big[\mathcal{K}(t)\big] = \frac{\mu+s^{\mu}(1-\mu)}{s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_0(\mu)\big(1+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}s^{-1}\big)}\Big(\big(\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)}\big)\mathcal{K}^0 + \mathscr{L}\Big[\xi\mathcal{J}(t)\mathcal{K}(t)-(\lambda+\phi)\mathcal{K}(t)\Big]\Big),\\ \mathscr{L}\big[\mathcal{L}(t)\big] = \frac{\mu+s^{\mu}(1-\mu)}{s^{\mu}\mathbb{AB}(\mu)\mathcal{A}_0(\mu)\big(1+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}s^{-1}\big)}\Big(\big(\frac{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)s^{\mu-1}}{\mu+s^{\mu}(1-\mu)}\big)\mathcal{L}^0 + \mathscr{L}\Big[\lambda\mathcal{K}(t)-(\eta+\phi)\mathcal{L}(t)\Big]\Big).\end{cases} \tag{71}$$

Equivalently

$$\begin{cases}\mathscr{L}[\mathcal{J}(t)] = \frac{\mathcal{J}^0}{s+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}} + \frac{\mu s^{-\mu-n}+s^{-n}(1-\mu)}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}\Big(\beta-\xi\mathcal{J}(t)\mathcal{K}(t)+\eta\mathcal{L}(t)-\phi\mathcal{J}(t)\Big),\\ \mathscr{L}[\mathcal{K}(t)] = \frac{\mathcal{K}^0}{s+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}} + \frac{\mu s^{-\mu-n}+s^{-n}(1-\mu)}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}\Big(\xi\mathcal{J}(t)\mathcal{K}(t)-(\lambda+\phi)\mathcal{K}(t)\Big),\\ \mathscr{L}[\mathcal{L}(t)] = \frac{\mathcal{L}^0}{s+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}} + \frac{\mu s^{-\mu-n}+s^{-n}(1-\mu)}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}\Big(\lambda\mathcal{K}(t)-(\eta+\phi)\mathcal{L}(t)\Big).\end{cases} \tag{72}$$

Consider equation (58) and (59), and using Inverse Laplace on both sides of equation (72) , we have

$$\begin{cases}\mathcal{J}_{k+1}(t) = \frac{1}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}^{-1}\Big[s^{-n}\Big(\mu s^{-\mu}+(1-\mu)\Big)\mathscr{L}\Big[-\xi\mathrm{Z}_k+\eta\mathcal{L}_k-\phi\mathcal{J}_k\Big]\Big],\\ \mathcal{K}_{k+1}(t) = \frac{1}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}^{-1}\Big[s^{-n}\Big(\mu s^{-\mu}+(1-\mu)\Big)\mathscr{L}\Big[\xi\mathrm{Z}_k-(\lambda+\phi)\mathcal{K}_k\Big]\Big],\\ \mathcal{L}_{k+1}(t) = \frac{1}{\mathbb{AB}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}^{-1}\Big[s^{-n}\Big(\mu s^{-\mu}+(1-\mu)\Big)\mathscr{L}\Big[\lambda\mathcal{K}_k-(\eta+\phi)\mathcal{L}_k\Big]\Big].\end{cases} \tag{73}$$

5. Transmission Dynamics of Skin Sores Disease Model with CPCF operator

Here we modify model (4) by replacing CPC with CPCF operator.

$$\begin{cases}{}_0^{CPCF}D_t^{\mu}J(t) = \beta-\xi J(t)K(t)+\eta L(t)-\phi J(t),\\ {}_0^{CPCF}D_t^{\mu}K(t) = \xi J(t)K(t)-(\lambda+\phi)K(t),\\ {}_0^{CPCF}D_t^{\mu}L(t) = \lambda K(t)-(\eta+\phi)L(t),\end{cases} \tag{74}$$

with non-negative initial values,

$$J(0)=J^0\ ,\ K(0)=K^0\ ,\ L(0)=L^0. \tag{75}$$

Theorem 5.1. *The Laplace Transform of CPCF operator is given [29] as:*

$$\begin{cases}\mathscr{L}[{}_0^{CPCF}D_t^{\mu}J(t)] = \Big(\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{\mu+s(1-\mu)}+\frac{s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\Big)\mathscr{L}[J(t)]-\Big(\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\Big)J^0,\\ \mathscr{L}[{}_0^{CPCF}D_t^{\mu}K(t)] = \Big(\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{\mu+s(1-\mu)}+\frac{s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\Big)\mathscr{L}[K(t)]-\Big(\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\Big)K^0,\\ \mathscr{L}[{}_0^{CPCF}D_t^{\mu}L(t)] = \Big(\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{\mu+s(1-\mu)}+\frac{s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\Big)\mathscr{L}[L(t)]-\Big(\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\Big)L^0.\end{cases} \tag{76}$$

Proof. Utilizing equation(3), we find

$$\begin{cases}\mathscr{L}[{}_0^{CPCF}D_t^{\mu}J(t)]=\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[J(t)]\mathscr{L}\left[\exp(-\frac{\mu}{1-\mu}t)\right]+\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\mathscr{L}[J'(t)]\mathscr{L}\left[\exp(-\frac{\mu}{1-\mu}t)\right],\\ \mathscr{L}[{}_0^{CPCF}D_t^{\mu}K(t)]=\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[K(t)]\mathscr{L}\left[\exp(-\frac{\mu}{1-\mu}t)\right]+\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\mathscr{L}[K'(t)]\mathscr{L}\left[\exp(-\frac{\mu}{1-\mu}t)\right],\\ \mathscr{L}[{}_0^{CPCF}D_t^{\mu}L(t)]=\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[L(t)]\mathscr{L}\left[\exp(-\frac{\mu}{1-\mu}t)\right]+\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\mathscr{L}[L'(t)]\mathscr{L}\left[\exp(-\frac{\mu}{1-\mu}t)\right].\end{cases} \tag{77}$$

$$\begin{cases}=\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[J(t)]\cdot\frac{1}{s+\frac{\mu}{1-\mu}}+\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\big(s\mathscr{L}[J(t)]-J(0)\big)\cdot\frac{1}{s+\frac{\mu}{1-\mu}},\\ =\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[K(t)]\cdot\frac{1}{s+\frac{\mu}{1-\mu}}+\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\big(s\mathscr{L}[K(t)]-K(0)\big)\cdot\frac{1}{s+\frac{\mu}{1-\mu}},\\ =\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{1-\mu}\mathscr{L}[L(t)]\cdot\frac{1}{s+\frac{\mu}{1-\mu}}+\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{1-\mu}\big(s\mathscr{L}[L(t)]-L(0)\big)\cdot\frac{1}{s+\frac{\mu}{1-\mu}}.\end{cases} \tag{78}$$

$$\begin{cases}=\big(\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{\mu+s(1-\mu)}\big)\mathscr{L}[J(t)]+\big(\frac{s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\big)\mathscr{L}[J(t)]-\big(\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\big)J^0,\\ =\big(\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{\mu+s(1-\mu)}\big)\mathscr{L}[K(t)]+\big(\frac{s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\big)\mathscr{L}[K(t)]-\big(\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\big)K^0,\\ =\big(\frac{\mathbb{M}(\mu)\mathcal{A}_1(\mu)}{\mu+s(1-\mu)}\big)\mathscr{L}[L(t)]+\big(\frac{s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\big)\mathscr{L}[L(t)]-\big(\frac{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}{\mu+s(1-\mu)}\big)L^0.\end{cases} \tag{79}$$

□

Theorem 5.2. *[29] Let*

$$\begin{cases}{}_0^{CPCF}D_t^{\mu}J(t)=\mathbb{S}_1(t),\\ {}_0^{CPCF}D_t^{\mu}K(t)=\mathbb{S}_2(t),\\ {}_0^{CPCF}D_t^{\mu}L(t)=\mathbb{S}_3(t).\end{cases} \tag{80}$$

Using Laplace Transform, we have

$$\begin{cases}J(t)=\frac{\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n{}_0I_t^{n+1}\mathbb{S}_1(t)+\frac{1-\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n{}_0I_t^{n}\mathbb{S}_1(t),\\ K(t)=\frac{\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n{}_0I_t^{n+1}\mathbb{S}_2(t)+\frac{1-\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n{}_0I_t^{n}\mathbb{S}_2(t),\\ L(t)=\frac{\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n{}_0I_t^{n+1}\mathbb{S}_3(t)+\frac{1-\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n{}_0I_t^{n}\mathbb{S}_3(t).\end{cases} \tag{81}$$

Proof. Consider class J from (80). As $\mathscr{L}[{}_0^{CPCF}D_t^{\mu}J(t)]=\mathscr{L}[\mathbb{S}_1(t)]$. Applying Theorem (5.1), we have

$$\mathscr{L}[J(t)]=\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)[\mathcal{A}_1(\mu)+s\mathcal{A}_0(\mu)]}\mathscr{L}[\mathbb{S}_1(t)]=\frac{\mu+s(1-\mu)}{s\mathbb{M}(\mu)\mathcal{A}_0(\mu)[1+\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}s^{-1}]}\mathscr{L}[\mathbb{S}_1(t)]. \tag{82}$$

After some simplification, we have

$$\begin{cases}\mathscr{L}[J(t)]&=\frac{\mu s^{-1}}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}s^{-1}\big)^n\mathscr{L}[\mathbb{S}_1(t)]+\frac{1-\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}s^{-1}\big)^n\mathscr{L}[\mathbb{S}_1(t)]\\ &=\Big(\frac{\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n s^{-n-1}+\frac{1-\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n s^{-n}\Big)\mathscr{L}[\mathbb{S}_1(t)].\end{cases} \tag{83}$$

Equivalently

$$J(t)=\frac{\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}[{}_0I_t^{n+1}\mathbb{S}_1(t)]+\frac{1-\mu}{\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\sum_{n=0}^{\infty}\big(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}\big)^n\mathscr{L}[{}_0I_t^{n}\mathbb{S}_1(t)]. \tag{84}$$

Using Inverse Laplace transform, we get the required result. □

Consider system (74) with non-negative initial values (75). Using Laplace Transform and implementing Theorem (5.1), we get the following recursive formula:

$$\begin{cases} J_{k+1}(t) = J^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left[\beta - \xi\mathscr{W}_k + \eta L_k - \phi J_k\right]\right), \\ K_{k+1}(t) = K^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left[\xi\mathscr{W}_k - (\lambda+\phi)K_k\right]\right), \\ L_{k+1}(t) = L^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left[\lambda K_k - (\eta+\phi)L_k\right]\right). \end{cases} \tag{85}$$

5.1. Analysis of Iteration Method

Theorem 5.3. *Let* $(\mathbf{G}, \|*\|)$ *be the Banach space, and let* $\mathbf{H}$ *be the self map of* $\mathbf{G}$ *that guarantees*

$$\|\mathbf{H}_i - \mathbf{H}_j\| \leq \mathbf{M}\|i - \mathbf{H}_i\| + m\|p-q\|, \tag{86}$$

for all $p,q \in \mathbf{G}$ *and* $0 \leq \mathbf{M}$*,* $0 \leq m \leq 1$.

Proof. Suppose

$$\begin{cases} J_{k+1}(t) = J^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left[\beta - \xi\mathscr{W}_k + \eta L_k - \phi J_k\right]\right), \\ K_{k+1}(t) = K^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left[\xi\mathscr{W}_k - (\lambda+\phi)K_k\right]\right), \\ L_{k+1}(t) = L^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left[\lambda K_k - (\eta+\phi)L_k\right]\right). \end{cases} \tag{87}$$

where $\frac{\mu+s(1-\mu)}{\mathbb{M}(\eta)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}$ denotes fractional Lagrange multiplier. □

Theorem 5.4. *Consider a self-map* $\mathbf{H}$ *described as*

$$\begin{cases} \mathbf{H}(J_k) = J_{k+1}(t) = J^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left[\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left(\beta - \xi J_k K_k + \eta L_k - \phi J_k\right)\right], \\ \mathbf{H}(K_k) = K_{k+1}(t) = K^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left[\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left(\xi J_k K_k - (\lambda+\phi)K_k\right)\right], \\ \mathbf{H}(L_k) = L_{k+1}(t) = L^0 \exp\left(-\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_0(\mu)}t\right) + \mathscr{L}^{-1}\left[\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\mathscr{L}\left(\lambda K_k - (\eta+\phi)L_k\right)\right]. \end{cases} \tag{88}$$

$$\text{is } \mathbf{H}-\text{stable in } \mathbb{L}^1(a,b) \text{ if } \quad \begin{cases} \left[\xi Q_1(\mu)\Gamma_2 + \xi\Gamma_1 Q_2(\mu) + \eta Q_3(\mu) + \phi Q_4(\mu)\right] < 1, \\ \left[\xi Q_5(\mu)\Gamma_2 + \xi\Gamma_1 Q_6(\mu) + (\lambda+\phi)Q_7(\mu)\right] < 1, \\ \left[\lambda Q_8(\mu) + (\eta+\phi)Q_9(\mu)\right] < 1. \end{cases} \tag{89}$$

Proof. Consider class J from (88). We find the following for every $(p,q) \in \mathbb{N}\times\mathbb{N}$.

$$\begin{cases} \mathbf{H}[J^p(t)] - \mathbf{H}[J^q(t)] \\ = \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\right)\left(\mathscr{L}[\beta - \xi J^p K^p + \eta L^p - \phi J^p] - \mathscr{L}[\beta - \xi J^q K^q + \eta L^q - \phi J^q]\right). \end{cases} \tag{90}$$

Without losing generality, and applying norm, we get

$$\begin{cases} \|\mathbf{H}(J^p(t)) - \mathbf{H}(J^q(t))\| \\ \quad = \left\|\mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\right)\left(\mathscr{L}[\beta - \xi J^p K^p + \eta L^p - \phi J^p] - \mathscr{L}[\beta - \xi J^q K^q + \eta L^q - \phi J^q]\right)\right\|. \end{cases} \tag{91}$$

Using triangle inequality and simplifying expressions, we have

$$\begin{cases} \|\mathbf{H}(J^p(t)) - \mathbf{H}(J^q(t))\| \leq \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\right) \\ \qquad \times\left\{\mathscr{L}\left(\|-\xi J^p(K^p - K^q\| + \|-\xi K^q(J^p - J^q)\| + \|\eta(L^p - L^q)\| + \|-\phi(J^p - J^q)\|\right)\right\}. \end{cases} \tag{92}$$

Since the results have similar role, we can write that

$$\|J^p(t) - J^q(t)\| = \|K^p(t) - K^q(t)\| = \|L^p(t) - L^q(t)\|. \tag{93}$$

$$\Longrightarrow \begin{cases} \|\mathbf{H}(J^p(t)) - \mathbf{H}(J^q(t))\| \leq \mathscr{L}^{-1}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\right) \\ \qquad\qquad \times\left\{\mathscr{L}\left(\|-\xi J^p(J^p-J^q\| + \|-\xi K^q(J^p-J^q)\| + \|\eta(J^p-J^q)\| + \|-\phi(J^p-J^q)\|\right)\right\}. \end{cases} \tag{94}$$

Similarly we can find this for remaining classes. As $\{J^p\}$, $\{K^p\}$, $\{L^p\}$, are positive and convergent sequences. Hence, we have for every t

$$\|J^p\| \leq \Gamma_1, \quad \|K^p\| \leq \Gamma_2, \quad \|L^p\| \leq \Gamma_3. \tag{95}$$

Hence, we find finally

$$\begin{cases} \|\mathbf{H}(J^p(t)) - \mathbf{H}(J^q(t))\| \leq \left[\xi Q_1(\mu)\Gamma_2 + \xi\Gamma_1 Q_2(\mu) + \eta Q_3(\mu) + \phi Q_4(\mu)\right], \\ \|\mathbf{H}(K^p(t)) - \mathbf{H}(K^q(t))\| \leq \left[\xi Q_5(\mu)\Gamma_2 + \xi\Gamma_1 Q_6(\mu) + (\lambda+\phi)Q_7(\mu)\right], \\ \|\mathbf{H}(L^p(t)) - \mathbf{H}(L^q(t))\| \leq \left[\lambda Q_8(\mu) + (\eta+\phi)Q_9(\mu)\right]. \end{cases} \tag{96}$$

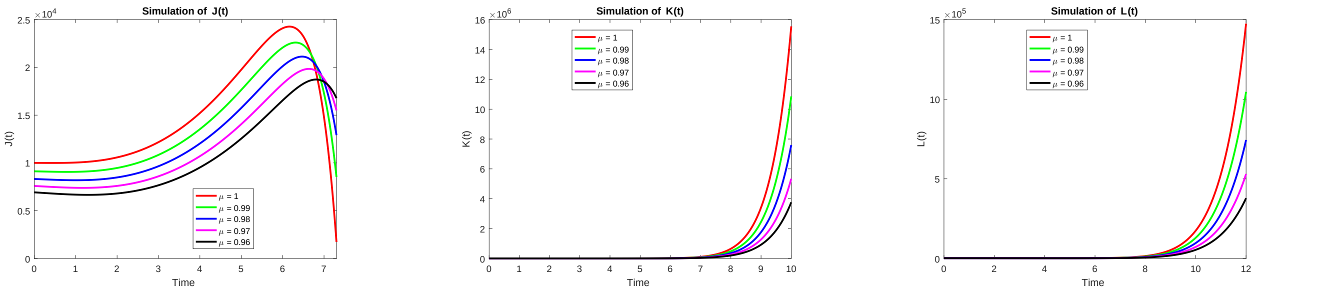
Here, $Q_i(\mu), i = 1,2,3,...,9$ represent functions from $\mathscr{L}^{-1}\left[\mathscr{L}\left(\frac{\mu+s(1-\mu)}{\mathbb{M}(\mu)\mathcal{A}_1(\mu)+s\mathbb{M}(\mu)\mathcal{A}_0(\mu)}\right)\right]$. Consequently, a fixed value exists in the mapping $\mathbf{H}$. We then conclude that $\mathbf{H}$ fulfils all the contexts in the aforementioned Theorem(5.3). Assume that equations (95) and (96) hold. This implies that $\Theta = (0,0,0)$. Where

$$\Theta = \begin{cases} \left[\xi Q_1(\mu)\Gamma_2 + \xi\Gamma_1 Q_2(\mu) + \eta Q_3(\mu) + \phi Q_4(\mu)\right] < 1, \\ \left[\xi Q_5(\mu)\Gamma_2 + \xi\Gamma_1 Q_6(\mu) + (\lambda+\phi)Q_7(\mu)\right] < 1, \\ \left[\lambda Q_8(\mu) + (\eta+\phi)Q_9(\mu)\right] < 1. \end{cases} \tag{97}$$

H ensures all the requirements of Theorem(5.4) and , is therefore, Picard **H**-stable. □

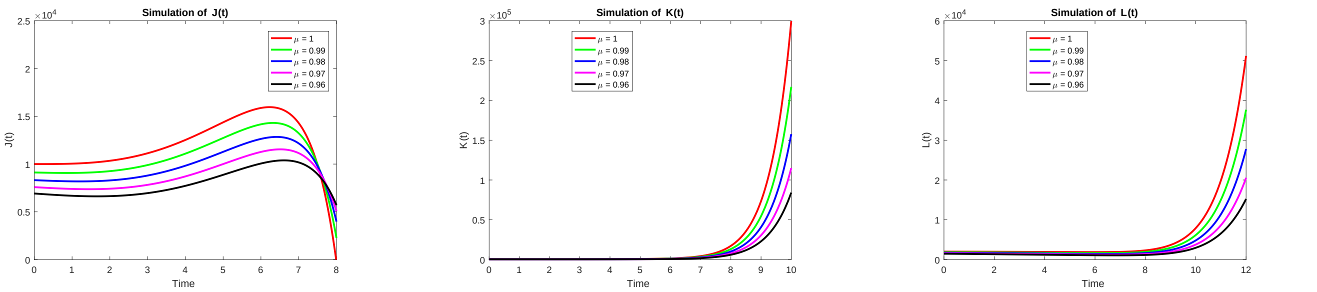
6. Result and Discussion

The fractional order skin sores model simulation is shown inside the pictures in this section using the following simple parameters from [3]: $\delta = 0.067$, $\lambda = 0.007$, $\eta = 0.00021$, $\phi = 0.012$, and $\beta = \delta\mathsf{N}$. These simulations show how models' behavior is impacted by changes in value. In order to give $\mathsf{N} = 12,500$ as the entire population under study, we have used initial points of $\mathsf{J}(0) = 10,000$, $\mathsf{K}(0) = 500$, and $\mathsf{L}(0) = 2000$. Figure (2) simulates the series results of equations (58) with CPC operator according to distinct values of μ, while Figure (3) simulates the series outcomes with CPCF operator based on distinguished values of μ, and Figure (4) simulates the series outcomes with CPCF operator according to distinct values of μ. MATLAB is used for simulating these recurring results. Figures (5), (6), and (7) also compare plots with CPC, CPABC, and CPCF operators for the simulations of J, K, and L with varying fractional order μ, $(\mu = 0.99, 0.98, 0.97)$. For this assessment. According to the study, ordinary derivatives have less degrees of freedom than the fractional order skin sores version. The compartments of the model use non-integer values of the fractional parameter to display first-rate feedback. Growth or lower interest actions happen more quickly in small fractional orders than in large fractional orders. It's obvious that this system's efficacy might be greatly increased by reducing the step time and calculating more terms or components for every variable. In presenting physical procedures, fractional order derivations have been shown to be more effective, dependable, and superior than classical orders. The results of the numerical simulation show that this method is a very effective and extreme way to find analytical solutions for a wide variety of fractional nonlinear systems.



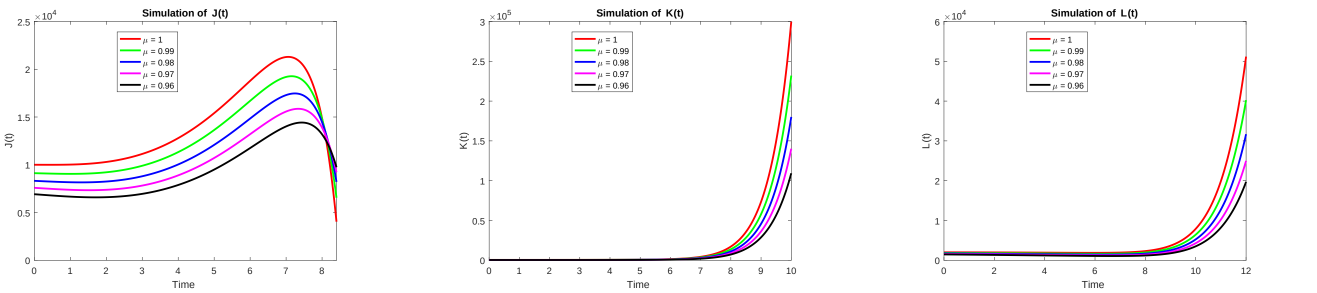
(a) Simulation of class $\mathbb{J}$ **(b)** Simulation of class $\mathbb{K}$ **(c)** Simulation of class $\mathbb{L}$

Figure 2: Simulation of Skin Sores model with CPC operator for various values of fractional order μ



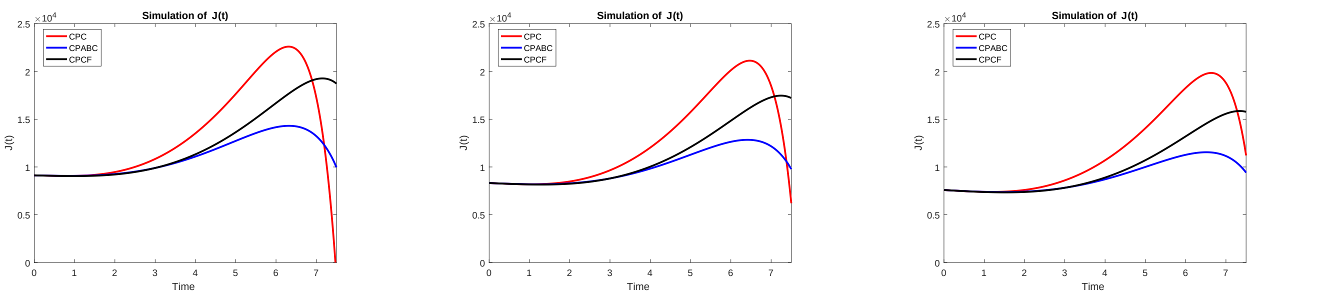
(a) Simulation of class $\mathcal{J}$ **(b)** Simulation of class $\mathcal{K}$ **(c)** Simulation of class $\mathcal{L}$

Figure 3: Simulation of Skin Sores model with CPABC operator for various values of fractional order μ



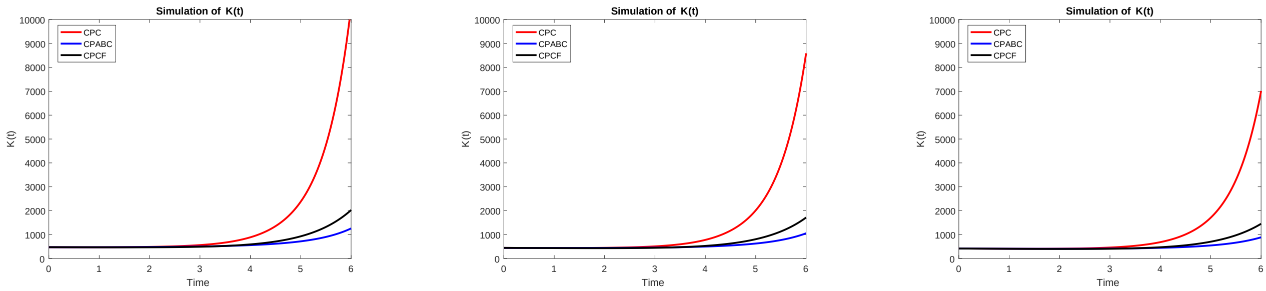
(a) Simulation of class J **(b)** Simulation of class K **(c)** Simulation of class L

Figure 4: Simulation of Skin Sores model with CPCF operator for various values of fractional order μ



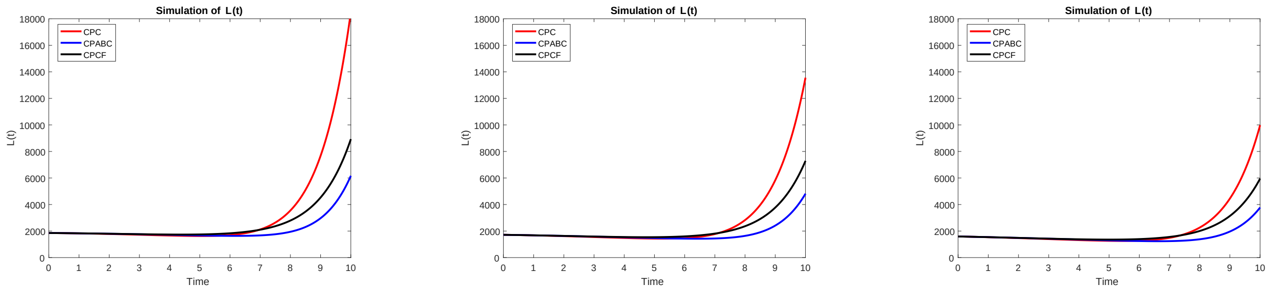
(a) $\mu = 0.99$ **(b)** $\mu = 0.98$ **(c)** $\mu = 0.97$

Figure 5: Simulation of Susceptible individuals for various values of fractional order μ



(a) $\mu = 0.99$ **(b)** $\mu = 0.98$ **(c)** $\mu = 0.97$

Figure 6: Simulation of Infected individuals for various values of fractional order μ



(a) $\mu = 0.99$ **(b)** $\mu = 0.98$ **(c)** $\mu = 0.97$

Figure 7: Simulation of Recovered individuals for various values of fractional order μ

7. Conclusion

An important issue in the applied sciences is examined in this paper: a nonlinear fractional order skin sores disease model. The study emphasizes how mathematical modeling can be used to prepare, negotiate, and manipulate the catastrophic social repercussions of infectious diseases. The study assesses the CPC, CPABC, and CPCF operators and discusses both qualitative and quantitative criticisms. In order to model the impacts of high fractional order and fractal size values, the authors additionally create a numerical simulation using MATLAB and LADM. One effective mathematical model for the disease of skin sores is the fractional operator. The fractional order can be changed to affect its behavior. The model is helpful and offers insight into infection dynamics because of its graphical effect. The fractional order derivation examines the full transmission from the infected character to the end, whereas the non-integer order derivation examines the sickness in a single area. In dermatology, this method enhances the effectiveness of traditional experimental methods. Policymakers and public health professionals may find the data useful in preventing the spread of skin sores through immunization.

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