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Dynamics of a Food Chain Model Using the Caputo Fractional Derivative

Aamir Shehzad*[a], Saba Jamil[a,b], Faiza Sarfraz[c]

[a]Mathematics Research Center, Near East University, Near East Boulevard, Nicosia North Cyprus, 99138
[b]Institute of Mathematics, kFUEIT, Rahim Yar Khan, Pakistan.
[c]Department of Mathematics, Lahore College for Women University, Lahore, Pakistan
Corresponding author Email:[aamir.shehzad.1054@gmail.com]

Abstract:

This work investigates a fractional-order food chain model based on the Caputo operator, taking into account resource availability, competition, and predation. The dynamics of the food chain are better represented using fractional calculus, which takes into account long-range interactions and previous dependency. Analytical and numerical simulations reveal information about the resilience, persistence, and stability of biological communities under fractional order dynamics. The model includes a three-species food chain, as well as environmental contamination. We begin by confirming the model's uniqueness, nonnegativity, and boundedness. We also examine numerous conditions for the presence of equilibrium and local stability. Second, a controller is proposed, and the global stability of the positive equilibrium point is investigated using the Lyapunov method. The proposed model solution was estimated using the fractional iterative technique, and numerical simulations were carried out to validate the theoretical results.

Keywords: Food chain model; Caputo fractional operator; Positivity and boundedness; Local stability; Chaos and error analysis.

1. Introduction

Environmental pollution is a global threat due to human activities like transportation, agriculture, urbanization, and industrialization. It affects ecosystem dynamics and biodiversity, with a focus on food chains and food webs. Understanding food chains helps depict the transfer of nutrients and energy across ecosystems, highlighting the interdependence of organisms. Pollution can disrupt food networks, disrupting survival, reproduction, and health. Population dynamics, including complex interspecific relationships like mutualism, parasitism, competition, and predation, are key areas of biomathematics study. Contaminants in land, water, and air can harm the biological chain, disrupt ecosystems, and lower biodiversity. Pollution negatively impacts wildlife populations, particularly marine animals and turtles. Water sources' turbidity ranges from 9.0 to 171.5, and air pollution negatively impacts wildlife health.

Researchers, in [1], investigated DNA methylation patterns and respiratory health of eastern gray squirrel populations in North Wales, Sussex, and London, focusing on ecological research relationships. Population dynamics are influenced by interactions in ecosystems, such as food chains and webs. Mathematical models using differential equations characterize these interactions, providing a foundation for ecological interaction networks. Research shows chaotic interactions between species in food chain, population, and predator-prey models, prompting scientists worldwide to use mathematical models to study ecological settings [2]. Li and Liu [3] investigated two biological species in a predator-prey paradigm, concentrating on how they moved in relation to the concentration of chemicals released by the other species. They imposed particular requirements on the initial data and framed the issue as a linked system. Garai et al. [4] investigated how predatory danger and prey protection affected predator-prey relationships., examining transition and chaos mechanisms and periodic structures in the ecological model. Researchers, in [5], studied a predator-prey biological system involving two species, revealing that over-predation of juveniles leads to system instability and supercritical Hoof bifurcations. In a predator-prey fisheries model, Sharif et al. [6] investigated the practice of fishing and developmental opportunities, analyzing homeostatic states and bifurcation results using numerical simulation analysis. Jiao et al. [7] developed a predator-prey model with impulses, analyzing its global asymptotic stability. The analysis revealed a continuous prey extinction boundary cycle solution, enriching population ecology and management. Das et al. [8] developed a predator-prey model to study natural interactions, including felt dread, interspecific rivalry, and delayed pregnancy in prey species. Pollution, particularly carbon dioxide emissions, significantly contributes to climate change, altering weather patterns and potentially affecting animal health. Reducing pollution is crucial for human survival, leading to increased interest in pollution management. To examine sulfur dioxide emissions and the existence of forests in various areas, Guo et al. [9] created an ecological model using the DPSIR structure, and [10] created a multiline atmospheric pollution prediction system with precise features.

In order to better explain difficult situations, fractional differential equationswhich include derivatives of unknown functions of fractional ordercan be used to describe problems in real life by offering comprehensive information (see [11][20]). It is more accurate and reliable to depict the actual dynamic process using fractional differential equations, as most mathematical models of biology have long-term memory. Fractional rabies and predator-prey models were put forth by [21], who also examined the stability, numerical solutions, and equilibrium points of each. The stability of a fractional-order system was investigated by the authors in [22] using the Lyapunov direct method, which significantly advanced methods for analyzing the systems' stability. The authors of [23] examined the dynamic behavior of the Hastings-Powell food chain system and extended it to fractional order. Using arbitrarily testing, contact monitoring, the use of condoms control parameters, and anti-retro-viral treatment (ART) to manage the epidemic, the authors of [24] created a fractional model for HIV/AIDS transmission dynamics. They discovered that ART treatment control for those with HIV who had been detected and monitoring their contacts greatly decreased the number of people who were still undetected. By simulating COVID-19 dynamics in diabetic patients, the model [25] offers comprehensive insights into the course of the disease. The progression of sickness is analyzed using fractional operators. Higher fatality rates among diabetic patients are revealed by the simulation, suggesting that these people require extra care. A SEACTR model for viral infection with hepatitis B, including treatment and vaccination controls, was proposed by Zehra et al. [26]. Dynamic behavior for constant controls was verified using the HBV model with a fractional operator. Lower fractional orders were found to improve HBV control in all models, and the study demonstrated good stability accuracy with smaller fractional orders. In order to improve accuracy and comprehension of the complexities of tuberculosis, a fractional operator was used in the development of a fractal-fractional model in [27] to explore population dynamics of the illness. To demonstrate how decreased forestry resources affect toxin activity and fire caused by humans, researchers [28] have created a nonlinear mathematical model using an improved ABC-fractional-order system. Additional research on fractional-order disease modeling can be found in [29, 30, 31]. Researchers have found that the system experiences bifurcation at a positive fixed point, and to mitigate this chaos, they employ various control strategies [32, 33].

This study investigates the application of a fractional model of species systems to accurately explain the dynamic behavior of multi-species food-chain ecosystems. Despite substantial research into biological models, omnivores are frequently disregarded in traditional models. The study looks at a food chain model in which herbivores, omnivores, and carnivores live while fully acknowledging their presence. This method is critical for comprehending the complexities of multispecies food-chain ecosystems. This work converts an integer-order food-chain system to fractional-order form, defining its order range, analyzing its dynamics with parameter value modification, and calculating its stability margin. It also investigates the dynamics of the resulting ecosystem. Next section covers some basic terms of fractional calculus used in this article. Section 3 suggests a fractional food chain model with a Holling type-II functional response based on these works. The existence, uniqueness, nonnegativity, and boundedness of the solution are all demonstrated in this study. The model's equilibrium point is computed and its local stability is established in the section 4. In the section 5, the suggested system of equations is subjected to a reliable numerical approach. Section 6 uses simulations to confirm the new model's validity by forecasting the food chain's dynamics. The conclusion is given in the section 7.

2. Preliminaries

Definition 2.1. *[16, 20, 40] For fractional order $\beta > 0$, the integral of a continuous function $f, \mathbb{R}^+ \to \mathbb{R}$ is given as:*

$$I_t^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau, \qquad t > 0,\ \beta \in (0,1). \tag{1}$$

Definition 2.2. *[16, 20, 40] The definition of a function $f, \mathbb{R}^+ \to \mathbb{R}$'s Caputo fractional derivative of order β is*

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\beta+1-n}} d\tau, \quad (n-1 < \beta < n), n \in \mathbb{R}^+. \tag{2}$$

At $n = 1$, we have

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\beta}} d\tau.$$

3. Model Formulation

In order to comprehend basic processes and preserve a sustainable environment, we investigate predator-prey interactions in ecosystems. Although functional responses are used in prey-predator and food chain models, the amount of time predators spend processing food is frequently ignored. In order to solve this problem and enhance prey capture, type II functional responses-which include a rectangular hyperbola-take into account predators' ability to absorb food. In 1889, H. Older and J. Hadamard created the general theory of convex functions, which is a useful tool for analyzing difficulties. Convex function-based inequalities are effective in many areas of mathematics and have drawn interest in literature. The model emphasizes the extensive effects that environmental contamination has on the ecosystem as a whole [34].

The population sizes are given as follows:

- $x(t)$: Prey population;
- $y(t)$: Intermediate predator predators;
- $z(t)$: Top predators.

Assumptions:

- We consider that logistical growth leads to a prey population.
- Let K represent the prey species' carrying capacity and r represent its intrinsic growth rate.
- To simulate the relationship between prey and intermediate predators, we have adapted the Holling-IV functional response [35].
- These three species are thought to reside in environments that are consistently polluted.
- Animal deaths can occur as a result of environmental pollution. Prey, intermediate predators, and top predators have pollution-related mortality rates of p_x, p_y, and p_z, in that order.

Biological systems can benefit from fractional order differential equations since they are more realistic because they can take into account local conditions and history. The prey predator food chain model is one example of the complex and nonlinear ecosystems that researchers are concentrating on using these models for. In an effort to increase the accuracy of their findings, they are also developing a number of ecological models with fractional order (see [36, 37, 38, 39]). Thus, we derive the following model based on the discussion above:

$$\begin{aligned}
{}^{C}_{0}D^{\beta}_{t}x(t) &= rx\left(1-\frac{x}{K}\right)-\frac{c_x xy}{ax^2+s}-p_x x,\\
{}^{C}_{0}D^{\beta}_{t}y(t) &= \frac{c_y xy}{ax^2+s}\cdot\frac{1+cy}{1+cy+wz}-\frac{c_z yz}{y+t}-d_y y-p_y y,\\
{}^{C}_{0}D^{\beta}_{t}z(t) &= \frac{c_u yz}{y+t}-d_z z-p_z z,
\end{aligned} \tag{3}$$

where the initial conditions are given as

$$x(0),\ y(0),\ z(0)\geq 0. \tag{4}$$

Table 1: Model parameters

Parameter	**Description**
r	Intrinsic growth rate.
K	Carrying capacity.
c_x	Attack rate by intermediate predators
c_y	Net gain of intermediate predators
p_x	Pollution-related death rate.
c_z	Attack rate by apex predators.
d_y	Natural death rate.
p_y	Pollution-related death rate.
c_u	Net gain over intermediate predators.
d_z	Natural death rate.
p_z	Pollution-related death rate.

3.1. Positivity and Uniqueness

Using a norm as a starting point, this section investigates constraints that guarantee positive, restricted, and well-posed solutions for the suggested model, assuming pertinent real-world conditions. The norm is

$$\|z\|_\infty = \sup_{t\in D_z}|z(t)|.$$

We find

$$\begin{aligned}
{}_0^C D_t^{\beta} x(t) &= rx\left(1-\frac{x}{K}\right) - \frac{c_x xy}{ax^2+s} - p_x x, \\
&\geq -(c_x|y| + p_x)x, \\
&\geq -(c_x \sup_{t\in D_{\mathbf{y}}} |y| + p_x)x, \\
&= -(c_x\|y\|_\infty + p_x)x, \\
\Rightarrow \quad & x(t) \geq x(0)\, e^{-(c_x\|y\|_\infty + p_x)t}, \forall t \geq 0. \qquad (5)
\end{aligned}$$

We can also have

$$y(t) \geq y(0)\, e^{-(c_z\|z\|_\infty + d_y + p_y)t}, \forall t \geq 0, \qquad (6)$$

$$z(t) \geq z(0)\, e^{-(c_u\|y\|_\infty + d_z + p_z)t}, \forall t \geq 0. \qquad (7)$$

Now, we examine that system solutions are limited in feasible region $\mathbb{R}^3_+$. On $(0,\infty)$, the system solutions are unique given to the initial conditions. We observe

- $x = 0$ suggests that ${}_0^C D_t^{\beta} x(t) = 0$,
- $y = 0$ suggests that ${}_0^C D_t^{\beta} y(t) = 0$,
- $z = 0$ suggests that ${}_0^C D_t^{\beta} z(t) = 0$,

The domain $\mathbb{R}^3_+$ is positive invariant, and each compartment has a unique solution, as seen by the above solutions' inability to escape the hyperplane.

4. Analyzing Equilibria and Local Stability

To obtain the Equilibrium points we let

$${}_0^C D_t^{\beta} x = {}_0^C D_t^{\beta} y = {}_0^C D_t^{\beta} z = 0,$$

and solve the system of equations simultaneously. Then, we get trivial and non trivial solutions which are as under;

$$E_1 = (x,y,z) = (0,0,0).$$

$$E_2 = (x,y,z) = \left(\frac{k(r-p_x)}{r}, 0, 0\right).$$

We now think about their stability. At the equilibrium point E_1, the Jacobian matrix is

$$J(E_1) = \begin{bmatrix} \frac{\partial f_1(x,y,z)}{\partial x} & \frac{\partial f_1(x,y,z)}{\partial y} & \frac{\partial f_1(x,y,z)}{\partial z}, \\ \frac{\partial f_2(x,y,z)}{\partial x} & \frac{\partial f_2(x,y,z)}{\partial y} & \frac{\partial f_2(x,y,z)}{\partial z}, \\ \frac{\partial f_3(x,y,z)}{\partial x} & \frac{\partial f_3(x,y,z)}{\partial y} & \frac{\partial f_3(x,y,z)}{\partial z}. \end{bmatrix} \qquad (8)$$

For Function $f_1(x,y,z)$:

$$\begin{aligned}
\frac{\partial}{\partial x}[-\frac{c_{\mathrm{x}}yx}{ax^2+s}+rx\cdot\left(1-\frac{x}{K}\right)-p_{\mathrm{x}}x] &= -c_x y\cdot\frac{\partial}{\partial x}\left[\frac{x}{ax^2+s}\right]+r\cdot\frac{\partial}{\partial x}\left[x\cdot\left(1-\frac{x}{K}\right)\right]-p_x\cdot\frac{\partial}{\partial x}[x],\\
&= -\frac{c_x y\cdot\left(1\left(ax^2+s\right)-\left(a\cdot\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}[s]\right)x\right)}{\left(ax^2+s\right)^2}+r\cdot\left(1\left(1-\frac{x}{K}\right)+x\cdot\left(\frac{d}{dx}[1]-\frac{1}{K}\cdot\frac{d}{dx}[x]\right)\right)-p_x,\\
&= -\frac{c_x y\cdot\left(ax^2-(a\cdot 2x+0)x+s\right)}{\left(ax^2+s\right)^2}+r\cdot\left(x\cdot\left(0-\frac{1}{K}\right)-\frac{x}{K}+1\right)-p_x,\\
&= -\frac{c_x y\cdot\left(s-ax^2\right)}{\left(ax^2+s\right)^2}+r\cdot\left(1-\frac{2x}{K}\right)-p_x,\\
&= -\frac{c_x y}{ax^2+s}+\frac{2ac_x yx^2}{\left(ax^2+s\right)^2}+r\cdot\left(1-\frac{x}{K}\right)-\frac{rx}{K}-p_x.
\end{aligned} \tag{9}$$

$$\begin{aligned}
\frac{\partial}{\partial y}\left[-\frac{c_{\mathrm{x}}xy}{ax^2+s}+rx\cdot\left(1-\frac{x}{k}\right)-p_{\mathrm{x}}x\right] &= -\frac{c_{\mathrm{x}}x}{ax^2+s}\cdot\frac{\partial}{\partial y}[y]+\frac{\partial}{\partial y}\left[rx\cdot\left(1-\frac{x}{k}\right)\right]+\frac{\partial}{\partial y}\left[-p_{\mathrm{x}}x\right],\\
&= -\frac{1c_{\mathrm{x}}x}{ax^2+s}+0+0, = -\frac{c_{\mathrm{x}}x}{ax^2+s}.
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{\partial}{\partial z}\left[-\frac{c_{\mathrm{x}}xy}{ax^2+s}+rx\cdot\left(1-\frac{x}{k}\right)-p_{\mathrm{x}}x\right] &= \frac{\partial}{\partial z}\left[-\frac{c_{\mathrm{x}}xy}{ax^2+s}\right]+\frac{\partial}{\partial z}\left[rx\cdot\left(1-\frac{x}{k}\right)\right]+\frac{\partial}{\partial z}\left[-p_{\mathrm{x}}x\right],\\
&= 0+0+0=0.
\end{aligned} \tag{11}$$

For Function $f_2(x,y,z)$:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\frac{c_{\mathrm{y}}y\cdot(cy+1)x}{(wz+cy+1)\left(ax^2+s\right)}-\frac{c_{\mathrm{z}}yz}{y+t}-p_{\mathrm{y}}y-d_{\mathrm{y}}y\right]\\
&\quad= \frac{c_{\mathrm{y}}y\cdot(cy+1)}{wz+cy+1}\cdot\frac{\partial}{\partial x}\left[\frac{x}{ax^2+s}\right]+\frac{\partial}{\partial x}\left[-\frac{c_{\mathrm{z}}yz}{y+t}\right]+\frac{\partial}{\partial x}\left[-p_{\mathrm{y}}y\right]+\frac{\partial}{\partial x}\left[-d_{\mathrm{y}}y\right],\\
&\quad= \frac{\frac{\frac{\partial}{\partial x}[x]\cdot\left(ax^2+s\right)-x\cdot\frac{\partial}{\partial x}\left[ax^2+s\right]}{\left(ax^2+s\right)^2}c_{\mathrm{y}}y\cdot(cy+1)}{wz+cy+1}+0+0+0,\\
&\quad= \frac{c_{\mathrm{y}}y\cdot(cy+1)\left(1\left(ax^2+s\right)-\left(a\cdot\frac{\partial}{\partial x}\left[x^2\right]+\frac{\partial}{\partial x}[s]\right)x\right)}{(wz+cy+1)\left(ax^2+s\right)^2},\\
&\quad= \frac{c_{\mathrm{y}}y\cdot(cy+1)\left(ax^2-(a\cdot 2x+0)x+s\right)}{(wz+cy+1)\left(ax^2+s\right)^2},\\
&\quad= \frac{c_{\mathrm{y}}y\cdot(cy+1)\left(s-ax^2\right)}{(wz+cy+1)\left(ax^2+s\right)^2}.
\end{aligned} \tag{12}$$

$$
\begin{aligned}
c\frac{\partial}{\partial y}&\Big[\frac{c_y xy\cdot(cy+1)}{(ax^2+s)\,(cy+wz+1)}-\frac{c_x zy}{y+t}-p_y y-d_y y\Big]\\
&=\frac{c_y x}{ax^2+s}\cdot\frac{d}{dy}\left[\frac{y\cdot(cy+1)}{cy+wz+1}\right]-c_z z\cdot\frac{\partial}{\partial y}\left[\frac{y}{y+t}\right]-p_y\cdot\frac{\partial}{\partial y}[y]-d_y\cdot\frac{\partial}{\partial y}[y],\\
&=\frac{\frac{\frac{\partial}{\partial y}\left[y\cdot(cy+1)\cdot(cy+wz+1)-y\cdot(cy+1)\cdot\frac{\partial}{\partial y}[cy+wz+1]\right.}{(cy+wz+1)^2}c_y x}{ax^2+s}-c_z z\cdot\frac{\frac{\partial}{\partial y}[y]\cdot(y+t)-y\cdot\frac{\partial}{\partial y}[y+t]}{(y+t)^2}-p_y\cdot 1-d_y\cdot 1,\\
&=-\frac{c_z tz}{(y+t)^2}+\frac{((y\cdot(c\cdot 1+0)+cy+1)(cy+wz+1)-cy\cdot(cy+1))c_y x}{(ax^2+s)\,(cy+wz+1)^2}-p_y-d_y,\\
&=-\frac{c_z tz}{(y+t)^2}+\frac{c_y x\cdot((cy+wz+1)(2cy+1)-cy\cdot(cy+1))}{(ax^2+s)\,(cy+wz+1)^2}-p_y-d_y.
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z}&\Big[\frac{c_y xy\cdot(cy+1)}{(ax^2+s)\,(wz+cy+1)}-\frac{c_z yz}{y+t}-p_y y-d_y y\Big]\\
&=\frac{c_y xy\cdot(cy+1)}{ax^2+s}\cdot\frac{\partial}{\partial z}\left[\frac{1}{wz+cy+1}\right]-\frac{c_z y}{y+t}\cdot\frac{\partial}{\partial z}[z]+\frac{\partial}{\partial z}\left[-p_y y\right]+\frac{\partial}{\partial z}\left[-d_y y\right],\\
&=-\frac{\frac{\frac{\partial}{\partial z}[wz+cy+1]}{(wz+cy+1)^2}c_y xy\cdot(cy+1)}{ax^2+s}-\frac{1c_z y}{y+t}+0+0,\\
&=-\frac{\left(w\cdot\frac{\partial}{\partial z}[z]+\frac{\partial}{\partial z}[cy]+\frac{\partial}{\partial z}[1]\right)c_y xy\cdot(cy+1)}{(ax^2+s)\,(wz+cy+1)^2}-\frac{c_z y}{y+t},\\
&=-\frac{(w\cdot 1+0+0)c_y xy\cdot(cy+1)}{(ax^2+s)\,(wz+cy+1)^2}-\frac{c_z y}{y+t},\\
&=-\frac{c_y wxy\cdot(cy+1)}{(ax^2+s)\,(wz+cy+1)^2}-\frac{c_z y}{y+t}.
\end{aligned}
\tag{14}
$$

For Function $f_3(x,y,z)$:

$$
\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{c_u yz}{y+t}-p_z z-d_z z\right]&=\frac{\partial}{\partial x}\left[\frac{c_u yz}{y+t}\right]+\frac{\partial}{\partial x}\left[-p_z z\right]+\frac{\partial}{\partial x}\left[-d_z z\right],\\
&=0+0+0,=0.
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y}\left[\frac{c_u zy}{y+t}-p_z z-d_z z\right]&=c_u z\cdot\frac{\partial}{\partial y}\left[\frac{y}{y+t}\right]+\frac{\partial}{\partial y}\left[-p_z z\right]+\frac{\partial}{\partial y}\left[-d_z z\right],\\
&=c_u z\cdot\frac{\frac{\partial}{\partial y}[y]\cdot(y+t)-y\cdot\frac{\partial}{\partial y}[y+t]}{(y+t)^2}+0+0,\\
&=\frac{c_u z\cdot\left(1(y+t)-\left(\frac{\partial}{\partial y}[y]+\frac{\partial}{\partial y}[t]\right)y\right)}{(y+t)^2},\\
&=\frac{c_u z\cdot(-(1+0)y+y+t)}{(y+t)^2},\\
&=\frac{c_u tz}{(y+t)^2}.
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z}\left[\frac{c_{\mathrm{u}}yz}{y+t}-p_{\mathrm{z}}z-d_{\mathrm{z}}z\right] &= \frac{c_{\mathrm{u}}y}{y+t}\cdot\frac{\partial}{\partial z}[z]-p_{\mathrm{z}}\cdot\frac{\partial}{\partial z}[z]-d_{\mathrm{z}}\cdot\frac{\partial}{\partial z}[z], \\
&= \frac{1c_{\mathrm{u}}y}{y+t}-p_{\mathrm{z}}\cdot 1-d_{\mathrm{z}}\cdot 1, \\
&= \frac{c_{\mathrm{u}}y}{y+t}-p_{\mathrm{z}}-d_{\mathrm{z}}.
\end{aligned} \tag{17}
$$

By using trivial equilibrium points $(E_1)=(x,y,z)=(0,0,0)$ in all above determined derivatives, we finally get in Jacobian at E_1 as:

$$
J(E_1)=\begin{bmatrix} r-P_x & 0 & 0 \\ 0 & -d_y-P_y & 0 \\ 0 & 0 & -d_z-P_z \end{bmatrix}. \tag{18}
$$

$$
|J(E_1)-\lambda I|=0 \;\Rightarrow\; \begin{bmatrix} r-p_x & 0 & 0 \\ 0 & -d_y-p_y & 0 \\ 0 & 0 & -d_z-p_z \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=0. \tag{19}
$$

$$
=\begin{vmatrix} -\lambda_1+(r-P_x) & 0 & 0 \\ 0 & -d_y-P_y-\lambda_2 & 0 \\ 0 & 0 & -\lambda_3-d_z-P_z \end{vmatrix}=0. \tag{20}
$$

It is obvious that

$$
\lambda_2<0 \text{ and } \lambda_3<0.
$$

Thus,

$$
\mathit{If}\,\lambda_1<0 \;\Rightarrow\; r-P_x<0\Rightarrow r<P_x. \tag{21}
$$

or $P_x>r$. Then, we can conclude that the trivial equilibrium $E_1=(0,0,0)$ is asymptotically stable. Now,By using non-trivial equilibrium points $E_2=(x,y,z)=\left(\frac{k(r-p_x)}{r},0,0\right)$ in all above pre determined derivatives, we get in Jacobian at E_2 as:

$$
J(E_2)=\begin{bmatrix} P_x-r & \frac{cxk(P_x-r)}{r\left(\frac{ak^2(P_x-r)^2}{r^2}+S\right)} & 0 \\ 0 & \frac{cyk(r-P_x)}{r\left(\frac{ak^2(r-P_x)^2}{r^2}+S\right)}-d_y-P_y & 0 \\ 0 & 0 & -d_z-P_z \end{bmatrix}. \tag{22}
$$

$$
|J(E_2)-\lambda I|=0 \Rightarrow \begin{bmatrix} P_x-r & \frac{c_xk(P_x-r)}{r\left(\frac{ak^2(P_x-r)^2}{r^2}+S\right)} & 0 \\ 0 & \frac{c_yk(r-P_x)}{r\left(\frac{ak^2(r-P_x)^2}{r^2}+s\right)} & 0 \\ 0 & 0 & -d_z-P_z \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=0. \tag{23}
$$

$$
\begin{bmatrix} -\lambda_1+(P_x-r) & \frac{c_xk(P_x-r)}{r\left(\frac{ak^2(P_x-r)}{r^2}+S\right)} & 0 \\ 0 & -\lambda_2+\frac{c_yk(r-P_x)}{r\left(\frac{\left(k^2(r-P_x)^2\right)}{r^2}+S\right)}-d_y-P_y & 0 \\ 0 & 0 & -\lambda+(-d_y-P_y) \end{bmatrix}=0. \tag{24}
$$

$$-\lambda_1 + (P_x - r) = 0.$$
$$-\lambda_2 + \frac{c_y k\,(r-P_x)}{r\left(\frac{ak^2(r-P_x)^2}{r^2}+S\right)} - d_y - P_y = 0. \tag{25}$$
$$-\lambda_3 + (d_z - P_z) = 0.$$

For the E_2, it is evident that $\lambda_1 < 0$ and $\lambda_3 = -dz - P_2 < 0$, E_2 is thus asymptotically stable when

$$\frac{\mathrm{C_y}\,k\,(r-P_x)}{r\left(\frac{ak^2(r-P_x)^2}{r^2}+S\right)} < d_y + P_y.$$

4.1. Chaos Control

The controlled systems with the functions $v_1(t)$, $v_2(t)$, and $v_3(t)$ are as follows:

$$\begin{aligned} \frac{d^\beta x}{dt^\beta} &= rx\left(1-\frac{x}{k}\right) - \frac{c_x xy}{ax^2+s} - P_x x + v_1(t), \\ \frac{d^\beta y}{dt^\beta} &= \frac{c_y xy}{ax^2+s}\cdot\frac{1+cy}{1+cy+\omega z} - \frac{c_z yz}{y+1} - d_y y - P_y y + v_2(t), \\ \frac{d^\beta z}{dt^\beta} &= \frac{c_u yz}{y+t} - d_z z - P_z z + v_3(t). \end{aligned} \tag{26}$$

where $(\bar{x}, \bar{y}, \bar{z})$ represent system's solutions. Consider the following error functions:

$$q_1 = x - \bar{x}, \quad q_2 = y - \bar{y} \quad \& q_3 = z - \bar{z}.$$

Thus, we can write

$$\begin{aligned} \frac{d^\beta q_1}{dt^\beta} &= rx\left(1-\frac{x}{k}\right) - \frac{c_x xy}{ax^2+s} - P_x x + v_1(t) - r\bar{x}\left(1-\frac{\bar{x}}{k}\right) - \frac{c_x\bar{x}\bar{y}}{a\bar{x}^2+s} - P_x\bar{x}, \\ \frac{d^\beta q_2}{dt^\beta} &= \frac{c_y xy}{ax^2+s}\cdot\frac{1+cy}{1+cy+wz} - \frac{c_z yz}{y+1} - d_y y - P_y y + v_2(t) - \frac{c_y\bar{x}\bar{y}}{a\bar{x}^2+s}\cdot\frac{1+c\bar{y}}{1+c\bar{y}+\omega\bar{z}} - \frac{c_x\bar{y}\bar{z}}{\bar{y}+1} - d_y\bar{y} - p_y\bar{y}, \\ \frac{d^\beta q_3}{dt^\beta} &= \frac{c_u yz}{y+t} - d_z z - P_z z + v_3(t) - \frac{c_u\bar{y}\bar{z}}{\bar{y}+t} - d_z\bar{z} - P_z\bar{z}. \end{aligned} \tag{27}$$

Theorem 4.1. *If the following control functions are taken into account, the error system approaches zero:*

$$\begin{cases} v_1(t) = -q_1 - \left[rx\left(1-\frac{x}{k}\right) - \frac{c_x xy}{ax^2+s} - p_x x\right] + \left[r\bar{x}\left(1-\frac{\bar{x}}{k}\right) - \frac{c_x\bar{x}\bar{y}}{a\bar{x}^2+s} - p_x\bar{x}\right], \\ v_2(t) = -\left[\frac{c_y xy}{ax^2+s}\cdot\frac{1+c_y}{1+cy+wz} - \frac{c_z yz}{y+1} - d_y y - P_y y\right] + \left[\frac{c_y\bar{x}\bar{y}}{a\bar{x}^2+s}\cdot\frac{1+c\bar{y}}{1+c\bar{y}+\omega\bar{z}} - \frac{c_x\bar{y}\bar{z}}{\bar{y}+1} - d_y\bar{y} - P_y\bar{y}\right], \\ v_3(t) = -\left[\frac{c_u yz}{y+t} - d_z z - p_z z\right] + \left[\frac{c_u\bar{y}\bar{z}}{\bar{y}+t} - d_z\bar{z} - P_z\bar{z}\right]. \end{cases} \tag{28}$$

Proof. The following Lyapunov function can be defined $H = \frac{1}{2}\left(q_1^2 + q_2^2 + q_3^2\right)$. According to the definition of a fractional derivative, taking the fractional derivative w.r.t H results in the following equations.

$$\frac{d^\beta H}{dt^\beta} \le q_1\frac{d^\beta q_1}{dt^\beta} + q_2\frac{d^\beta q_2}{dt^\beta} + q_3\frac{d^\beta q_3}{dt^\beta}. \tag{29}$$

Hence, the following equation results in:

$$\frac{d^\beta H}{dt^\beta} \le -q_1^2 - \delta_1 q_2^2 - \delta_2 q_3^2 < 0. \tag{30}$$

□

5. Numerical scheme

The food-chain problem can be discretized using a numerical technique [41] based on Newton polynomial interpolation. Let the system is as follows:

$$\begin{aligned}{}_0^C D_t^{\beta} x(t) &= \mathscr{K}_1(t,\omega(t)),\\ {}_0^C D_t^{\beta} y(t) &= \mathscr{K}_2(t,\omega(t)),\\ {}_0^C D_t^{\beta} z(t) &= \mathscr{K}_3(t,\omega(t)).\end{aligned} \tag{31}$$

where $\omega=(x,y,z)$, and

$$\begin{aligned}\mathscr{K}_1(t,\omega(t)) &= rx\left(1-\frac{x}{K}\right)-\frac{c_x xy}{ax^2+s}-p_x x,\\ \mathscr{K}_2(t,\omega(t)) &= \frac{c_y xy}{ax^2+s}\cdot\frac{1+cy}{1+cy+wz}-\frac{c_z yz}{y+t}-d_y y-p_y y,\\ \mathscr{K}_3(t,\omega(t)) &= \frac{c_u yz}{y+t}-d_z z-p_z z.\end{aligned} \tag{32}$$

When, we generalize the system as:

$$\begin{cases}{}_0^C D_t^{\beta}\omega(t)=\mathscr{K}_i(t,\omega(t)), & i=(1,2,3),\\ \omega(0)=\omega_0,\end{cases} \tag{33}$$

Rearrange in the following way:

$$\omega(t)-\omega(0)=\frac{1-\beta}{\Gamma(\beta)}\int_0^t \mathscr{K}_1(\tau,\omega(\tau))(t-\tau)^{\beta-1}d\tau. \tag{34}$$

We can write the following at point $\mathrm{t}_{t+1}=(t+1)\Delta\mathrm{t}$:

$$\omega\left(\mathrm{t}_{l+1}\right)=\omega(0)+\frac{1-\beta}{\Gamma(\beta)}\sum_{\ell=2}^{l}\int_{\mathrm{t}_\ell}^{\mathrm{t}_{\ell+1}}\mathscr{K}_1(\tau,\omega(\tau))\left(\mathrm{t}_{l+1}-\tau\right)^{\beta-1}\,\mathrm{d}\tau. \tag{35}$$

By substituting Newton polynomial, we obtain

$$\omega^{1+1}=\omega_0+\tfrac{1-\beta}{\Gamma(\beta)}\textstyle\sum_{\ell=2}^{l}\int_{\mathrm{t}_\ell}^{\mathrm{t}_{\ell+1}}\left\{\begin{array}{l}\mathscr{K}_1\left(\mathrm{t}_{\ell-2},\omega^{\ell-2}\right)\\ +\frac{\mathscr{K}_1\left(\mathrm{t}_{\ell-1},\omega^{\ell-1}\right)-\mathscr{K}_1\left(\mathrm{t}_{\ell-2},\omega^{\ell-2}\right)}{\Delta\mathrm{t}}\left(\tau-\mathrm{t}_{\ell-2}\right)\\ +\frac{\mathscr{K}_1\left(\mathrm{t}\ell,\omega^{\ell}\right)-2\mathscr{K}_1\left(\mathrm{t}_{\ell-1},\omega^{\ell-1}\right)\left(\mathscr{K}_1\left(\mathrm{t}_{\ell-2},\omega^{\ell-2}\right)\right)}{2(\Delta\mathrm{t})^2}\\ \times\left(\tau-\mathrm{t}_{\ell-2}\right)\left(\tau-\mathrm{t}_{\ell-1}\right)\end{array}\right\}\times\left(\mathrm{t}_{\ell+1}-\tau\right)^{l-1}\mathrm{d}\tau. \tag{36}$$

As a result of various calculations, we can see the following strategy:

$$
\begin{aligned}
\omega^{l+1} &= \omega_0 + \frac{(\Delta \mathrm{t})^{\beta}}{\Gamma(\beta+1)} \sum_{\ell=2}^{l} \mathscr{K}_1\left(\mathrm{t}_{\ell-2}, \omega^{\ell-2}\right) \times \left[(t-\ell+1)^{\beta} - (t-\ell)^{\beta}\right] \\
&+ \frac{(\Delta \mathrm{t})^{\beta}}{\Gamma(\beta+2)} \sum_{\ell=2}^{l} \left[\mathscr{K}_1\left(\mathrm{t}_{\ell-1}, \omega^{\ell-1}\right) - \mathscr{K}_1\left(\mathrm{t}_{\ell-2}, \omega^{\ell-2}\right)\right] \\
&\times \left[\begin{array}{c} (t-\ell+1)^{\beta}(t-\ell+3+2\beta) \\ -(t-\ell)^{\beta}(t-\ell+3+3\beta) \end{array}\right] \\
&+ \frac{(\Delta \mathrm{t})^{\beta}}{2\Gamma(\beta+3)} \sum_{\ell=2}^{l} \left[\begin{array}{r} \mathscr{K}_1\left(\mathrm{t}\ell, \omega^{\ell}\right) - 2\mathscr{K}_1\left(\mathrm{t}_{\ell-1}, \omega^{\ell-1}\right) \\ +\mathscr{K}_1\left(\mathrm{t}_{\ell-2}, \omega^{\ell-2}\right) \end{array}\right] \\
&\times \left[\begin{array}{c} (l-\ell+1)^{\beta} \left\{\begin{array}{l} 2(t-\ell)^2 \\ +(3\beta+10)(t-\ell) \\ +2\beta^2+9\beta+12 \end{array}\right\} \\ -(t-\ell)^{\beta} \left\{\begin{array}{l} 2(l-\ell)^2 \\ +5(\beta+2)(t-\ell) \\ +6(\beta^2+3\beta+2) \end{array}\right\} \end{array}\right].
\end{aligned} \tag{37}
$$

6. Results and Discussions

This part provides validation for the numerical scheme presented in the previous section. To validate and authenticate the results, the food chain model under the Caputo operator is numerically simulated using MATLAB, which is also utilized to simulate the analytical results. For the evolution of the proposed food web model, some initial conditions and fractional order β are provided. Parameters' values are assumed and adjusted accordingly. The three compartments of the suggested food-chain model are visually depicted in numerical simulations on various fractional orders (β).

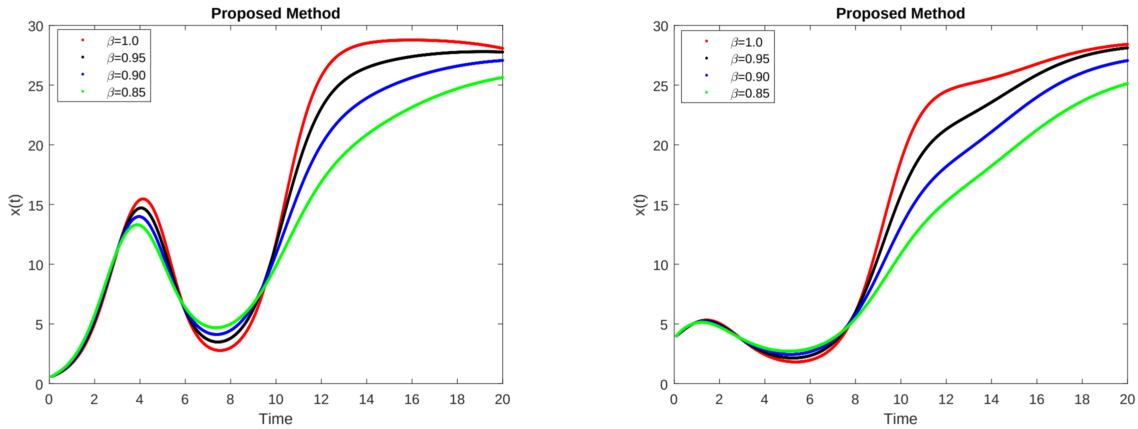
- The $x(t)$ compartment simulation, depicted in Figure (1), reveals dynamic variations at different fractional orders. Moderate variations occur with fractional order 0.85, indicating slower response to input stimuli. As fractional order approaches 0.95, behavior intensifies, possibly causing faster reactions. Larger fractional orders may amplify oscillations or accelerate system reactions. Initial conditions and transient behaviors significantly impact trajectory and steady-state responses, highlighting the impact of initial points and transient behaviors.

- According to Figure (2), the compartment $y(t)$ can be simulated to understand its behavior under different fractional orders and starting conditions. At fractional order 0.6, $y(t)$ responds slowly, while at fractional order 0.95, it behaves more dynamically. The change in initial conditions from (0.6, 3.1, 5) to (4, 10, 3.2) highlights the impact of starting conditions on $y(t)$'s initial behavior and steady-state response.

- The simulation findings reveal that compartment z(t) behaves differently under different fractional orders and conditions, as shown in Figure (3). Higher fractional orders, like 0.95, produce more noticeable oscillations or dynamic behavior, while lower fractional orders, like 0.85, provide sluggish dynamics. The change in initial conditions from 0.6 to 3.2 demonstrates how simulation's beginning points affect $z(t)$'s transient behavior and final steady-state response.

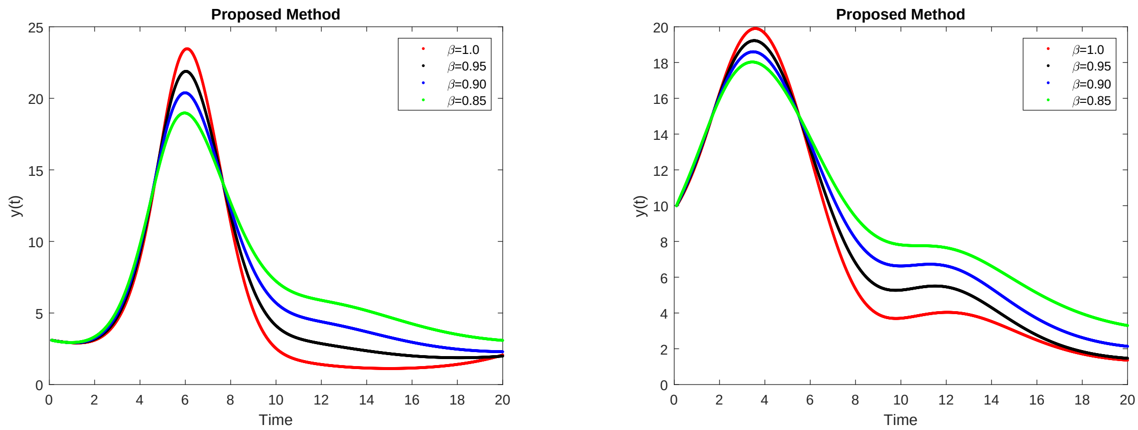
- The fractional order significantly influences system behavior, with higher orders indicating faster or more dynamic responses. The initial conditions chosen significantly influence the final steady-state response and transient behavior of all three compartments.

- Because it is non-local and simultaneously examines fractional operator and fractal dimension, the Caputo derivative performs better than classical derivatives. Over longer time periods, fractional-order model solutions exhibit a slower rise and decrease.



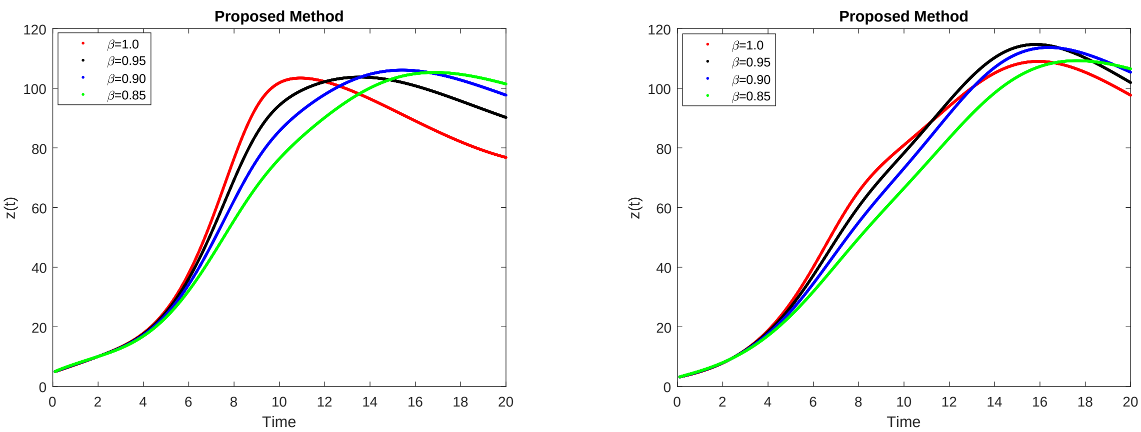
(a) (x(0),y(0),z(0))=(0.6,3.1,5) **(b)** (x(0),y(0),z(0))=(4, 10, 3.2)

Figure 1: Simulation of x(t) at different β values and different initial conditions



(a) (x(0),y(0),z(0))=(0.6,3.1,5) **(b)** (x(0),y(0),z(0))=(4, 10, 3.2)

Figure 2: Simulation of y(t) at different β values and different initial conditions



(a) (x(0),y(0),z(0))=(0.6,3.1,5) **(b)** (x(0),y(0),z(0))=(4, 10, 3.2)

Figure 3: Simulation of z(t) at different β values and different initial conditions

Comprehending these dynamics is essential for improving strategies and guaranteeing reliable operation in diverse settings. Additionally, fractional order dynamics must be taken into account while modeling and controlling tasks in order to appropriately depict system behavior. Researchers and practitioners can change β values to model a variety of events, from complicated non-linear dynamics to exponential decay.

7. Conclusion

This paper examines equilibrium points and stability features in a three-species Caputo fractional-order food chain model. It demonstrated the boundedness of solutions and validated the existence of unique solutions. The study focused on chaos management techniques and aimed to control and stabilize chaotic processes within a system using numerical methods and graphical representations, providing informative visualizations. The approach provides information on the prey-predator food chain in an ecosystem by demonstrating stable conditions for suggested species on numerous fractional orders (β). Stable conditions for species on various fractional orders and fractal dimensions are provided by the intricate geometrical analysis. Each species' reliance on its surroundings is depicted graphically, exposing chaotic behavior. Because quantities are displayed in spectral format, behavior between 0 and 1 can be examined. Among other aspects, the dynamic analysis of time-delay fractional-order systems and control methods need further investigation. Using numerical simulations and theory analysis, the stability margin of a regulated fractional-order ecosystem is found.

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