Journal of Mathematical Modeling and Fractional Calculus
https://acspublisher.com/journals/index.php/jmmfc
Received:24-01-2024, **Revised:** 26-02-2024,
Accepted:02-03-2024, **Published:** ,
DOI: 1(1),25-37.

Hybrid Fractional Operator for Epidemic Model analysis and application

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Abstract: The Constant proportional Caputo (CPC) fractional derivative has been one of the most useful operators for modeling non-local behaviors by fractional differential equations. We proposed the CPC operator for the drinking epidemic which is the world wide largest issue nowadays. In comparison to the integer-order models, the fractional order differential equations models appear to be more compatible with this illness. Qualitative and quantitative analysis for model and scheme are treated. We also present the uniqueness and Ulam-Hyres stability of solutions to a particular class of fractional starting value issues involving the Hilfer proportional fractional derivative using certain well-known theorems from the fixed point theory. The bivariate Mittag-Leffler function that was recently published will be used to solve the drinking pandemic model, we first construct the inverse operator and Laplace transform of the new formulation also eigenfunctions for the proposed scheme. Some important properties were also verified for the Constant proportional Caputo (CPC) fractional derivative on the epidermic model.
Keywords: Drinking Model; Stability ; Uniqueness; Constant proportional Caputo; Eigen functions

1. Introduction

Alcohol increases the production of insulin, which speeds up glucose metabolism and can cause low blood sugar, which can make diabetics irritable and even cause their death. Teenagers who still have growing brains are more susceptible to having an alcohol consumption disorder. Teenagers who drink are more likely to suffer harm, even death. The majority of practical mathematics is devoted to the study of differential equations and their solutions. A differential equation, either ordinary or partial, can be used to model almost any dynamic process in nature. The monograph that AA Kilbas worked on offers the most recent and current research on fractional differential and fractional integro-differential equations using a variety of potentially practical fractional calculus operators. Calculus of integrals and derivatives of any arbitrary real or complex order is the topic of fractional calculus and its applications [1]. On, Y Zhang worked Many models have still to be proposed, explored, and used to practical applications in many disciplines of science and engineering where nonlocality plays a significant part in fractional calculus. Although many fantastic discoveries have previously been reported by researchers in important monographs and review articles, there are still a great deal of non-local phenomena that have not been studied and are only waiting to be found [2]. Baleanu provided a derivative with fractional order to find the important questions while Ali Akgul worked on Atangana. In this research, we introduce a new method for studying fractional differential equations, including the fractional derivative of the Atangana-Baleanu problem [3]. The primary objective of EK Akgl's study is to use the Mittag-Leffler nonsingular kernel to solve linear and nonlinear fractional differential equations. To solve this issue, a precise numerical approach has been developed. Two experiments are used to support the theoretical findings [4]. The derivatives are understood in the Caputo meaning by NJ Ford et al. the existence and uniqueness of solutions are discussed analytically first, and then we look at how the solutions relate to the available information [5]. D. Baleanu et al. adapt and tweak the method to solve a large class of partial differential equations of fractional order as required. We demonstrate the approach's value by applying it to the solution of a model fractional issue [6].

After low birth weight and risky sex, alcohol ranks third globally in terms of causes of illness and early mortality. A novel fractional derivative with a non-local and non-singular kernel was proposed by D. Baleanu et al. We discussed some of the new derivative's beneficial characteristics and used it to resolve the fractional heat transfer model [7]. For the majority of enterprises in Kenya, a Stephen et al. [8] considerable prevalence of alcohol issues in the workforce has been a growing source of concern. Due to issues related to alcoholism, the majority of employees exhibit erratic work attendance, low productivity, bad health, and safety hazards. A more realistic binge drinking model with time delay was developed by Huo HF et al. In our approach, the time lag of the immunity against drinking is represented by time delay. Routh-Hurwitz criteria for the model without a time delay [9]. Despite significant advancements in oncological outcomes for patients with rectal cancer over the past few decades, M Grade et al. found that high levels of impairment persisted in anorectal, urinary, and sexual functions regardless of whether radical surgery was carried out open or laparoscopically [10]. Neurophysiological studies of 100 long-term alcoholics who were receiving neuropsychiatric care by D. Mller et colleagues. revealed signs of polytopic damage to the peripheral and central nervous systems. The findings demonstrate the need for a thorough diagnostic programme in order to identify the damage [11]. The fractional-order COVID-19 model is examined using the Atangana-Baleanu-Caputo fractional derivative and the Omicron effect's deterministic mathematical model is examined using various fractional parameters [12]. The following proportional derivative operator was defined in [15]:

$$ {}^{P}D_t^{\upsilon}\psi(t) = \mathbb{M}_1(\upsilon,t)\psi(t) + \mathbb{M}_0(\upsilon,t)\psi'(t) \tag{1} $$

Where ψ is a differentiable function of $t \in \mathbf{R}$ and $\mathbb{M}_1$ and $\mathbb{M}_0$ are functions of $\upsilon \in [0,1]$ and t in $t \in \mathbf{R}$ meeting certain requirements. This operator naturally occurs in the context of control theory and is related to the broad and evolving concept of conformable derivatives.

2. The Hybrid Fractional Derivative Operator

We obtained various useful and substantial nonlinear dynamics and contemporary calculus findings.

Definition 2.1. *Recall that [13] defined the Caputo derivative of a differentiable function $\psi(t)$ to order $\upsilon \in (0,1)$ with beginning point $t = 0$ as follows:*

$$ {}^{C}D_t^{\upsilon}\psi(t) = \frac{1}{\Gamma(1-\upsilon)}\int_0^t \psi'(\rho)(t-\rho)^{-\upsilon}d\rho \tag{2} $$

Definition 2.2. *Let $\psi(t)$ be an integrable function, $0 < \upsilon < 1$, then the RiemannLiouville integral is defined as follows [16]:*

$$ {}^{RL}D_t^{\upsilon}\psi(t) = \frac{1}{\Gamma(\upsilon)}\int_0^t (t-\rho)^{\upsilon-1}\psi(\rho)d\rho \tag{3} $$

Definition 2.3. *The new type of fractional operator is defined as a hybrid fractional operator from combining the proportional and Caputo definition [16]:*

$$ {}^{CP}D_t^{\upsilon}\psi(t) = \frac{1}{\Gamma(\upsilon)}\int_0^t (t-\rho)^{-\upsilon}\left(\mathbb{M}_1(\upsilon,t)\psi(t) + \mathbb{M}_0(\upsilon,t)\psi'(t)\right)d\rho \tag{4} $$

Definition 2.4. *One of the common methods for extending the Riemann-Liouville integral, which is defined by, is the equation (2).*

$$^{RL}I_t^{\upsilon}\psi(t) = \frac{1}{\Gamma(\upsilon)}\int_0^t \psi(\rho)(t-\rho)^{\upsilon-1}d\rho \tag{5}$$

given an integrable function $\psi(t)$ and $\upsilon > 0$. The definitions make it quite evident that the Caputo derivative

$$^{C}D_t^{\upsilon}\psi(t) =^{RL} I_t^{1-\upsilon}\psi(t) \tag{6}$$

which as a definition of fractional derivatives makes some sense. The Caputo derivative also has the following other well-known characteristics [14]:

$$^{RL}I_t^{\upsilon C}D_t^{\upsilon}\psi(t) = \psi(t) - \psi(0)$$

$$^{C}D_t^{\upsilon RL}I_t^{\upsilon}\psi(t) = \psi(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)}\lim_{t\to 0}{}^{RL}I_t^{\upsilon}\psi(0)$$

$$\mathbb{L}\left[{}^{C}D_t^{\upsilon}\psi(t)\right] = S^{\upsilon}\mathbb{L}[\psi(t)] - S^{\upsilon-1}\psi(0)$$

where the symbol $\mathbb{L}$ denotes the Laplace transformation from a function of t to a function of S. We highlight these characteristics since they will be crucial in showing outcomes regarding our new operators in the future.

Definition 2.5. *We recall the following universal non-fractional differential operator, often known as a "proportional" or "conformable" differential operator, from [15]:*

$$^{P}D_t^{\upsilon}\psi(t) = \mathbb{M}_1(\upsilon,t)\psi(t) + \mathbb{M}_0(\upsilon,t)\psi'(t) \tag{7}$$

where the functions of the variables t and $\upsilon \in [0,1]$ are $\mathbb{M}_1$ and $\mathbb{M}_0$, respectively, and they meet the following criteria $\forall t \in \mathbf{R}$:

$$\lim_{\upsilon\to 0^+}\mathbb{M}_0(\upsilon,t) = 0; \quad \lim_{\upsilon\to 1^-}\mathbb{M}_0(\upsilon,t) = 1; \quad \mathbb{M}_0(\upsilon,t) \neq 0, \upsilon \in (0,1] \tag{8}$$

$$\lim_{\upsilon\to 0^+}\mathbb{M}_1(\upsilon,t) = 1; \quad \lim_{\upsilon\to 1^-}\mathbb{M}_1(\upsilon,t) = 0; \quad \mathbb{M}_1(\upsilon,t) \neq 0, \upsilon \in [0,1) \tag{9}$$

The usual differentiation operator $D\psi(t) = \psi'(t)$, which depends on the arbitrary parameter υ, can be understood as being generalized by this.

3. Mathematical Model

Significant ingredients of the model are detached here in the given points.

3.1. Model description

The population is divided into four divisions according to our mathematical model, which is displayed in tabale1

Table 1: The proposed model's classes are explained.

$\mathbf{S}(t)$	Class of Non-Drinkers
$\mathbf{H}(t)$	Class of Heavy Drinkers
$\mathbf{T}(t)$	Class of Drinkers in Treatment
$\mathbf{R}(t)$	Class of Recovered Drinkers

3.2. Model assumptions

The following assumptions were made in the model:

1. The drinking pandemic takes place in a restricted setting.
2. The likelihood of becoming a heavy drinker is unaffected by a person's gender, race, or socioeconomic class.
3. When non-drinkers come into touch with heavy drinkers, heavy drinking is transferred to them.
4. Members who interact uniformly have a same degree of mixing.
5. Only once a patient has gone through the recovery and vulnerable compartments can they resume excessive drinking.
6. People who have given up drinking join the rehabilitation area.

4. Fractional order model of Drinking Epidemic Model

We present a deterministic compartmental model of drinking epidemic transmission dynamics [17]'s compartmental mathematical epidemic model

$$\begin{cases} {}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{S}(t) = b - \alpha\mathbf{SH} - \mu\mathbf{S} + \eta\mathbf{R} \\ {}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{H}(t) = \alpha\mathbf{SH} - (\mu + \delta_1 + \phi)\mathbf{H} \\ {}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{T}(t) = \phi\mathbf{H} - (\mu + \delta_2 + \gamma)\mathbf{T} \\ {}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{R}(t) = \gamma\mathbf{T} - (\mu + \eta)\mathbf{R} \end{cases} \tag{10}$$

$$\mathbf{S}(0) = \mathbf{S}^0, \mathbf{H}(0) = \mathbf{H}^0, \mathbf{T}(0) = \mathbf{T}^0, \mathbf{R}(0) = \mathbf{R}^0 \tag{11}$$

The total population is thus determined by:

$$\mathbb{N}(t) = \mathbf{S}(t) + \mathbf{H}(t) + \mathbf{T}(t) + \mathbf{R}(t) \tag{12}$$

4.1. Uniqueness Result

The uniqueness of the suggested problem's solutions (10) will be thoroughly demonstrated in this paragraph utilizing the ideas from the Banach contraction principle [20]. We thus require the following presumptions.

1. Let $\mathbf{S} : \mathrm{J} \times \mathrm{R} \to \mathrm{R}$, $\mathbf{H} : \mathrm{J} \times \mathrm{R} \to \mathrm{R}$, $\mathbf{T} : \mathrm{J} \times \mathrm{R} \to \mathrm{R}$ and $\mathbf{R} : \mathrm{J} \times \mathrm{R} \to \mathrm{R}$ be a function such that $\mathbf{S} \in \mathscr{C}_{1-\beta}^{q(1-P)}[\mathrm{J},\mathrm{R}]$, $\mathbf{H} \in \mathscr{C}_{1-\beta}^{q(1-P)}[\mathrm{J},\mathrm{R}]$, $\mathbf{T} \in \mathscr{C}_{1-\beta}^{q(1-P)}[\mathrm{J},\mathrm{R}]$ and $\mathbf{R} \in \mathscr{C}_{1-\beta}^{q(1-P)}[\mathrm{J},\mathrm{R}]$ for any $t \in \mathscr{C}_{1-\beta}^{\beta}[\mathrm{J},\mathrm{R}]$
2. There exists a constant κ such that

$$\begin{cases} |\mathbf{S}(t,v) - \mathbf{S}(t,\overline{v})| \leq \kappa|v - \overline{v}|, \\ |\mathbf{H}(t,v) - \mathbf{H}(t,\overline{v})| \leq \kappa|v - \overline{v}|, \\ |\mathbf{T}(t,v) - \mathbf{T}(t,\overline{v})| \leq \kappa|v - \overline{v}|, \\ |\mathbf{R}(t,v) - \mathbf{R}(t,\overline{v})| \leq \kappa|v - \overline{v}|. \end{cases}$$

   for any $v, \overline{v} \in \mathrm{R}$ and $t \in \mathrm{J}$
3. Suppose that $\kappa\Omega < 1$

where

$$\Omega = \frac{\mathfrak{B}(\beta,P)}{\rho^P\Gamma(P)}\left\{|\Lambda|\sum_{n=1}^{m} c_n(t_n - a)^{P+\beta-1} + (T-a)^P\right\}$$

and $\mathfrak{B}(\beta,P)$ is the Beta function defined by [20]

$$\mathfrak{B}(\beta,P) = \int_0^1 t^{\beta-1}(1-t)^{P-1}dt \qquad\qquad Re(\beta), Re(P) > 0$$

Theorem 4.1. *Let $0 < P < 1$, $0 \leq q \leq 1$ and $\beta = P + q - Pq$. Now we suppose that the assumptions 1-3 are satisfied. Then problem (10) has a unique solution in the space $\mathscr{C}_{1-\beta}^{\beta}[\mathrm{J},\mathrm{R}]$.*

Proof. Define the operator $\mathbb{T} : \mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}] \to \mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]$ by

$$\begin{cases} \mathbb{T}y(t) = \frac{\Lambda}{\upsilon^P\Gamma(P)} e^{\frac{(\upsilon-1)}{\upsilon}(t-a)^{\beta-1}} \sum_{n=1}^{m} \int_a^{t_n} e^{\frac{(\upsilon-1)}{\upsilon}(t_n-\rho)^{P-1}} \mathbf{S}(\rho,y(\rho))d\rho \\ +\frac{1}{\upsilon^P\Gamma(P)} \int_a^t e^{\frac{(\upsilon-1)}{\upsilon}(t-\rho)^{P-1}} \mathbf{S}(\rho,y(\rho))d\rho, \\ \mathbb{T}y(t) = \frac{\Lambda}{\upsilon^P\Gamma(P)} e^{\frac{(\upsilon-1)}{\upsilon}(t-a)^{\beta-1}} \sum_{n=1}^{m} \int_a^{t_n} e^{\frac{(\upsilon-1)}{\upsilon}(t_n-\rho)^{P-1}} \mathbf{H}(\rho,y(\rho))d\rho \\ +\frac{1}{\upsilon^P\Gamma(P)} \int_a^t e^{\frac{(\upsilon-1)}{\upsilon}(t-\rho)^{P-1}} \mathbf{H}(\rho,y(\rho))d\rho, \\ \mathbb{T}y(t) = \frac{\Lambda}{\upsilon^P\Gamma(P)} e^{\frac{(\upsilon-1)}{\upsilon}(t-a)^{\beta-1}} \sum_{n=1}^{m} \int_a^{t_n} e^{\frac{(\upsilon-1)}{\upsilon}(t_n-\rho)^{P-1}} \mathbf{T}(\rho,y(\rho))d\rho \\ +\frac{1}{\upsilon^P\Gamma(P)} \int_a^t e^{\frac{(\upsilon-1)}{\upsilon}(t-\rho)^{P-1}} \mathbf{T}(\rho,y(\rho))d\rho, \\ \mathbb{T}y(t) = \frac{\Lambda}{\upsilon^P\Gamma(P)} e^{\frac{(\upsilon-1)}{\upsilon}(t-a)^{\beta-1}} \sum_{n=1}^{m} \int_a^{t_n} e^{\frac{(\upsilon-1)}{\upsilon}(t_n-\rho)^{P-1}} \mathbf{R}(\rho,y(\rho))d\rho \\ +\frac{1}{\upsilon^P\Gamma(P)} \int_a^t e^{\frac{(\upsilon-1)}{\upsilon}(t-\rho)^{P-1}} \mathbf{R}(\rho,y(\rho))d\rho. \end{cases}$$

$$\begin{cases} \leq \frac{\kappa|\Lambda|}{\upsilon^P\Gamma(P)} \mathfrak{B}(\beta,P) \sum_{n=1}^{m} c_n(t_n-\rho)^{P+\beta-1} \|y_1-y_2\|\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}] \\ +\frac{\kappa}{\upsilon^P\Gamma(P)}(T-a)^P \mathfrak{B}(\beta,P)\|y_1-y_2\|\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}], \\ \leq \frac{\kappa|\Lambda|}{\upsilon^P\Gamma(P)} \mathfrak{B}(\beta,P) \sum_{n=1}^{m} c_n(t_n-\rho)^{P+\beta-1} \|y_1-y_2\|\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}] \\ +\frac{\kappa}{\upsilon^P\Gamma(P)}(T-a)^P \mathfrak{B}(\beta,P)\|y_1-y_2\|\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}], \\ \leq \frac{\kappa|\Lambda|}{\upsilon^P\Gamma(P)} \mathfrak{B}(\beta,P) \sum_{n=1}^{m} c_n(t_n-\rho)^{P+\beta-1} \|y_1-y_2\|\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}] \\ +\frac{\kappa}{\upsilon^P\Gamma(P)}(T-a)^P \mathfrak{B}(\beta,P)\|y_1-y_2\|\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}], \\ \leq \frac{\kappa|\Lambda|}{\upsilon^P\Gamma(P)} \mathfrak{B}(\beta,P) \sum_{n=1}^{m} c_n(t_n-\rho)^{P+\beta-1} \|y_1-y_2\|\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}] \\ +\frac{\kappa}{\upsilon^P\Gamma(P)}(T-a)^P \mathfrak{B}(\beta,P)\|y_1-y_2\|\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]. \end{cases}$$

Therefore,

$$\begin{cases} \|(Ty_1)-(Ty_2)\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]} \leq \frac{\kappa}{\upsilon^P\Gamma(P)} \mathfrak{B}(\beta,P) \left\{ |\Lambda| \sum_{n=1}^{m} c_n(t_n-\rho)^{P+\beta-1} + (T-a)^P \right\} \|y_1-y_2\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]}, \\ \|(Ty_1)-(Ty_2)\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]} \leq \frac{\kappa}{\upsilon^P\Gamma(P)} \mathfrak{B}(\beta,P) \left\{ |\Lambda| \sum_{n=1}^{m} c_n(t_n-\rho)^{P+\beta-1} + (T-a)^P \right\} \|y_1-y_2\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]}, \\ \|(Ty_1)-(Ty_2)\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]} \leq \frac{\kappa}{\upsilon^P\Gamma(P)} \mathfrak{B}(\beta,P) \left\{ |\Lambda| \sum_{n=1}^{m} c_n(t_n-\rho)^{P+\beta-1} + (T-a)^P \right\} \|y_1-y_2\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]}, \\ \|(Ty_1)-(Ty_2)\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]} \leq \frac{\kappa}{\upsilon^P\Gamma(P)} \mathfrak{B}(\beta,P) \left\{ |\Lambda| \sum_{n=1}^{m} c_n(t_n-\rho)^{P+\beta-1} + (T-a)^P \right\} \|y_1-y_2\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]}. \end{cases}$$

$$\begin{cases} \leq \kappa\Phi\|y_1-y_2\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]}, \\ \leq \kappa\Phi\|y_1-y_2\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]}, \\ \leq \kappa\Phi\|y_1-y_2\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]}, \\ \leq \kappa\Phi\|y_1-y_2\|_{\mathscr{C}_{1-\beta}[\mathrm{J},\mathrm{R}]}. \end{cases}$$

The conclusion that T is a contraction map derives from the aforementioned premise. Because of the Banach contraction principle, the suggested model (10) has a singular solution. □

4.2. Ulam-Hyres stability

In this section, we demonstrate the demonstration [22, 23]'s Ulam-Hyres stability.

Definition 4.1. *If for any positive ε and for all $\Phi \in (\mathrm{W}[0,T],\mathrm{R})$ there exists a positive operator R_{a_1,a_2} then system (10)is Ulam Hyres stable*

$$|{}^{CPC}D_t^{\upsilon_1,\upsilon_2}\Phi(t) - \Omega(t,\widehat{\Phi(t)})| \leq \varepsilon \qquad \forall t \in [0,T]$$

so we get a special outcome $\Upsilon \in (\mathrm{W}[0,T],\mathrm{R})$ *such that*

$$|\Phi(t)-\Upsilon(t)| \leq \mathrm{R}_{a_1,a_2}\varepsilon \qquad \forall t \in [0,T]$$

If we suppose a perturbation $\Psi \in \mathrm{W}[0,T]$ then $\Psi(0)=0$. Now consider

- For $\varepsilon > 0$, we have $|\Psi(t)| \leq \varepsilon$
- ${}^{CPC}D_t^{\upsilon_1,\upsilon_2}\Phi(t) = \Upsilon(t,\widehat{\Phi(t)})+\Psi(t)$

Lemma 4.2. *A perturbed model has outcome*

$$ {}^{CPC}D_t^{a_1,a_2}\Phi(t) = \Upsilon(t,\widehat{\Phi(t)})+\Psi(t) \qquad \Phi(0)=\Phi_0$$

fulfills the following relation

$$\left|\mathrm{R}(t)-\left\{\Phi(0)+\frac{a_2t^{a_2-1}(1-a_1)}{AB(a_1)}\Upsilon(t,\Phi(t))+\frac{a_1a_2}{AB(a_1)\Gamma(a_1)}\int_0^t \lambda^{a_2-1}(t-\lambda)^{a_2-1}\Upsilon(\lambda,\Phi(\lambda))d\lambda\right\}\right| \leq \chi^*_{a_1,a_2}\varepsilon$$

$$\chi^*_{a_1,a_2} = \frac{a_2T^{a_2-1}(1-a_1)}{ab(a_1)}+\frac{a_1a_2}{AB(a_1)\Gamma(a_1)}T^{a_1+a_2-1}H(a_1,a_2)$$

Lemma 4.3. *If we consider the condition of with lemma (4.2) then it has Ulam-Hyres stable solution under the condition* $\rho < 1$.

Proof. Consider $\alpha \in A$ be a special result, and $\Phi \in A$ be any result of the model, then

$$\begin{aligned}|\Phi(t)-\alpha(t)| = \Big|\Phi(t)-\Big\{\alpha(0)&+\frac{a_2t^{a_1-1}(1-a_1)}{AB(a_1)}\Upsilon(t,\alpha(t))\\ &+\frac{a_1a_2}{AB(a_1)\Gamma(a_1)}\int_0^t \lambda^{a_2-1}(t-\lambda)^{a_2-1}\Upsilon(\lambda,\alpha(\lambda))d\lambda\Big\}\Big|\end{aligned}$$

$$\begin{aligned}&\leq \left|\Phi(t)-\left\{\alpha(0)+\frac{a_2t^{a_2-1}(1-a_1)}{AB(a_1)}\Upsilon(t,\Phi(t))+\frac{a_1a_2}{AB(a_1)\Gamma(a_1)}\int_0^t \lambda^{a_2-1}(t-\lambda)^{a_2-1}\Upsilon(\lambda,\Phi(\lambda))d\lambda\right\}\right|\\
&\leq \left|\Phi(0)+\frac{a_2t^{a_2-1}(1-a_1)}{AB(a_1)}\Upsilon(t,\Phi(t))+\frac{a_1a_2}{AB(a_1)\Gamma(a_1)}\int_0^t \lambda^{a_2-1}(1-\lambda)^{a_2-1}\Upsilon(\lambda,\Phi(\lambda))d\lambda\right|\\
&-\left|\alpha(0)+\frac{a_2t^{a_2-1}(1-a_1)}{AB(a_1)}\Upsilon(t,\alpha(t))+\frac{a_1a_2}{AB(a_1)\Gamma(a_1)}\int_0^t \lambda^{a_2-1}(1-\lambda)^{a_2-1}\Upsilon(\lambda,\alpha(\lambda))d\lambda\right|\\
&\leq \chi_{a_1,a_2}\varepsilon+\left\{\frac{a_2T^{a_2-1}(1-a_1)}{AB(a_1)}+fraca_1a_2AB(a_1)\Gamma(a_1)T^{a_1+a_2-1}H(a_1,a_2)\right\}L_\Upsilon\Big|\Phi(t)-\alpha(t)\Big|\\
&\leq \chi_{a_1,a_2}\varepsilon+\rho|\Phi(t)-\alpha(t)|\end{aligned}$$

Consequently,

$$\|\Phi-\alpha\| \leq \chi_{a_1,a_2}\varepsilon+\rho|\Phi(t)-\alpha(t)|$$

Moreover, we can write the above expression as

$$\|\Phi-\alpha\| \leq \mathrm{R}_{a_1,a_2}\varepsilon$$

where $\mathrm{R}_{a_1,a_2} = \frac{\chi_{a_1,a_2}}{1-\rho}$. Hence it is Ulam-Hyres stable. □

5. Fractional Integral Operator of Proposed Model

5.1. Inverting by Operational Calculus

Since a Riemann-Liouville fractional integral with proportional derivatives, which gives both the PC and CPC fractional operators, is composed of

$$ {}^{PC}D_t^{\upsilon}\mathbf{S}(t) = {}^{RL}\mathbb{I}_t^{1-\upsilon}\left[{}^{P}D_t^{\upsilon}\mathbf{S}(t)\right], \quad and \quad {}^{CPC}D_t^{\upsilon}\mathbf{S}(t) = {}^{RL}\mathbb{I}_t^{1-\upsilon}\left[{}^{PC}D_t^{\upsilon}\mathbf{S}(t)\right] \tag{13} $$

$$ {}^{PC}D_t^{\upsilon}\mathbf{H}(t) = {}^{RL}\mathbb{I}_t^{1-\upsilon}\left[{}^{P}D_t^{\upsilon}\mathbf{H}(t)\right], \quad and \quad {}^{CPC}D_t^{\upsilon}\mathbf{H}(t) = {}^{RL}\mathbb{I}_t^{1-\upsilon}\left[{}^{PC}D_t^{\upsilon}\mathbf{H}(t)\right] \tag{14} $$

$$ {}^{PC}D_t^{\upsilon}\mathbf{T}(t) = {}^{RL}\mathbb{I}_t^{1-\upsilon}\left[{}^{P}D_t^{\upsilon}\mathbf{T}(t)\right], \quad and \quad {}^{CPC}D_t^{\upsilon}\mathbf{T}(t) = {}^{RL}\mathbb{I}_t^{1-\upsilon}\left[{}^{PC}D_t^{\upsilon}\mathbf{T}(t)\right] \tag{15} $$

$$ {}^{PC}D_t^{\upsilon}\mathbf{R}(t) = {}^{RL}\mathbb{I}_t^{1-\upsilon}\left[{}^{P}D_t^{\upsilon}\mathbf{R}(t)\right], \quad and \quad {}^{CPC}D_t^{\upsilon}\mathbf{R}(t) = {}^{RL}\mathbb{I}_t^{1-\upsilon}\left[{}^{PC}D_t^{\upsilon}\mathbf{R}(t)\right] \tag{16} $$

it follows that to invert the fractional operators it will be sufficient to invert both the RiemannLiouville integral and the proportional derivatives ${}^{P}D_t^{\upsilon}$ and ${}^{PC}D_t^{\upsilon}$. The Riemann-Liouville derivative inverts the Riemann-Liouville integral, and [17] developed the inverse of the proportional derivative. In the subsequent Lemma, we present the latter outcome.

Lemma 5.1. *The expression for the inverse of the proportional derivative operator ${}^{P}D_t^{\upsilon}$ is*

$$ \begin{cases} {}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \int_a^t exp\left[-\int_u^t \frac{\mathbb{M}_1(\upsilon,S)}{\mathbb{M}_0(\upsilon,S)}dS\right]\frac{\mathbf{S}(u)}{\mathbb{M}_0(\upsilon,u)}du \\ {}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \int_a^t exp\left[-\int_u^t \frac{\mathbb{M}_3(\upsilon,S)}{\mathbb{M}_2(\upsilon,S)}dS\right]\frac{\mathbf{H}(u)}{\mathbb{M}_2(\upsilon,u)}du \\ {}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \int_a^t exp\left[-\int_u^t \frac{\mathbb{M}_5(\upsilon,S)}{\mathbb{M}_4(\upsilon,S)}dS\right]\frac{\mathbf{T}(u)}{\mathbb{M}_4(\upsilon,u)}du \\ {}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \int_a^t exp\left[-\int_u^t \frac{\mathbb{M}_7(\upsilon,S)}{\mathbb{M}_6(\upsilon,S)}dS\right]\frac{\mathbf{R}(u)}{\mathbb{M}_6(\upsilon,u)}du \end{cases} \tag{17} $$

and this satisfies the following inversion relations:

$$ \begin{cases} {}^{P}\mathbb{D}_t^{\upsilon}{}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \mathbf{S}(t), & {}^{P}\mathbb{D}_t^{\upsilon}{}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \mathbf{S}(t) - exp\left[-\int_a^t \frac{\mathbb{M}_1(\upsilon,S)}{\mathbb{M}_0(\upsilon,S)}dS\right]\mathbf{S}(0) \\ {}^{P}\mathbb{D}_t^{\upsilon}{}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \mathbf{H}(t), & {}^{P}\mathbb{D}_t^{\upsilon}{}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \mathbf{H}(t) - exp\left[-\int_a^t \frac{\mathbb{M}_3(\upsilon,S)}{\mathbb{M}_2(\upsilon,S)}dS\right]\mathbf{H}(0) \\ {}^{P}\mathbb{D}_t^{\upsilon}{}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \mathbf{T}(t), & {}^{P}\mathbb{D}_t^{\upsilon}{}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \mathbf{T}(t) - exp\left[-\int_a^t \frac{\mathbb{M}_5(\upsilon,S)}{\mathbb{M}_4(\upsilon,S)}dS\right]\mathbf{T}(0) \\ {}^{P}\mathbb{D}_t^{\upsilon}{}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \mathbf{R}(t), & {}^{P}\mathbb{D}_t^{\upsilon}{}^{P}_{a}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \mathbf{R}(t) - exp\left[-\int_a^t \frac{\mathbb{M}_7(\upsilon,S)}{\mathbb{M}_6(\upsilon,S)}dS\right]\mathbf{R}(0) \end{cases} \tag{18} $$

In particular, for the constant-coefficient operator $PC\mathbb{D}_t^{\upsilon}$, the integral formula is

$$ \begin{cases} {}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \frac{1}{\mathbb{M}_0(\upsilon)}\int_a^t exp\left[-\frac{\mathbb{M}_1(\upsilon)}{\mathbb{M}_0(\upsilon)}(t-u)\right]\mathbf{S}(u)du, \\ {}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \frac{1}{\mathbb{M}_2(\upsilon)}\int_a^t exp\left[-\frac{\mathbb{M}_3(\upsilon)}{\mathbb{M}_2(\upsilon)}(t-u)\right]\mathbf{H}(u)du, \\ {}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \frac{1}{\mathbb{M}_4(\upsilon)}\int_a^t exp\left[-\frac{\mathbb{M}_5(\upsilon)}{\mathbb{M}_4(\upsilon)}(t-u)\right]\mathbf{T}(u)du, \\ {}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \frac{1}{\mathbb{M}_6(\upsilon)}\int_a^t exp\left[-\frac{\mathbb{M}_7(\upsilon)}{\mathbb{M}_6(\upsilon)}(t-u)\right]\mathbf{R}(u)du. \end{cases} \tag{19} $$

and the inversion relations are

$$ \begin{cases} {}^{PC}\mathbb{D}_t^{\upsilon}{}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \mathbf{S}(t), & {}^{PC}\mathbb{D}_t^{\upsilon}{}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \mathbf{S}(t) - exp\left[-\frac{\mathbb{M}_1(\upsilon)}{\mathbb{M}_0(\upsilon)}(t-a)\right]\mathbf{S}(a), \\ {}^{PC}\mathbb{D}_t^{\upsilon}{}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \mathbf{H}(t), & {}^{PC}\mathbb{D}_t^{\upsilon}{}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \mathbf{H}(t) - exp\left[-\frac{\mathbb{M}_3(\upsilon)}{\mathbb{M}_2(\upsilon)}(t-a)\right]\mathbf{H}(a), \\ {}^{PC}\mathbb{D}_t^{\upsilon}{}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \mathbf{T}(t), & {}^{PC}\mathbb{D}_t^{\upsilon}{}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \mathbf{T}(t) - exp\left[-\frac{\mathbb{M}_5(\upsilon)}{\mathbb{M}_4(\upsilon)}(t-a)\right]\mathbf{T}(a), \\ {}^{PC}\mathbb{D}_t^{\upsilon}{}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \mathbf{R}(t), & {}^{PC}\mathbb{D}_t^{\upsilon}{}^{PC}_{a}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \mathbf{R}(t) - exp\left[-\frac{\mathbb{M}_7(\upsilon)}{\mathbb{M}_6(\upsilon)}(t-a)\right]\mathbf{R}(a). \end{cases} \tag{20} $$

Note that, if $\mathbf{S}(a) = 0$, $\mathbf{H}(a) = 0$, $\mathbf{T}(a) = 0$, $\mathbf{R}(a) = 0$ then the operators ${}^{P}\mathbb{D}_t^{\upsilon}$, ${}^{P}_{a}\mathbb{I}_t^{\upsilon}$ and ${}^{PC}\mathbb{D}_t^{\upsilon}$, ${}^{PC}_{a}\mathbb{I}_t^{\upsilon}$ generate two-sided inverse pairings of one another.

Proposition 6.1 The inverse operators to the fractional PC and CPC derivatives are given by (??)-(??).

$$\begin{cases} {}^{PC}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \int_0^t exp\left[-\int_u^t \frac{\mathbb{M}_1(\upsilon,S)}{\mathbb{M}_0(\upsilon,S)}dS\right] \frac{{}^{RL}\mathbb{D}_u^{1-\upsilon}\mathbf{S}(u)}{\mathbb{M}_0(\upsilon,u)}du, \\ {}^{PC}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \int_0^t exp\left[-\int_u^t \frac{\mathbb{M}_3(\upsilon,S)}{\mathbb{M}_2(\upsilon,S)}dS\right] \frac{{}^{RL}\mathbb{D}_u^{1-\upsilon}\mathbf{H}(u)}{\mathbb{M}_2(\upsilon,u)}du, \\ {}^{PC}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \int_0^t exp\left[-\int_u^t \frac{\mathbb{M}_5(\upsilon,S)}{\mathbb{M}_4(\upsilon,S)}dS\right] \frac{{}^{RL}\mathbb{D}_u^{1-\upsilon}\mathbf{T}(u)}{\mathbb{M}_4(\upsilon,u)}du, \\ {}^{PC}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \int_0^t exp\left[-\int_u^t \frac{\mathbb{M}_7(\upsilon,S)}{\mathbb{M}_6(\upsilon,S)}dS\right] \frac{{}^{RL}\mathbb{D}_u^{1-\upsilon}\mathbf{R}(u)}{\mathbb{M}_6(\upsilon,u)}du. \end{cases} \tag{21}$$

$$\begin{cases} {}^{CPC}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \frac{1}{\mathbb{M}_0(\upsilon)} \int_0^t exp\left[-\frac{\mathbb{M}_1(\upsilon)}{\mathbb{M}_0(\upsilon)}(t-u)\right] {}^{RL}\mathbb{D}_u^{1-\upsilon}\mathbf{S}(u)du, \\ {}^{CPC}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \frac{1}{\mathbb{M}_2(\upsilon)} \int_0^t exp\left[-\frac{\mathbb{M}_3(\upsilon)}{\mathbb{M}_2(\upsilon)}(t-u)\right] {}^{RL}\mathbb{D}_u^{1-\upsilon}\mathbf{H}(u)du, \\ {}^{CPC}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \frac{1}{\mathbb{M}_4(\upsilon)} \int_0^t exp\left[-\frac{\mathbb{M}_5(\upsilon)}{\mathbb{M}_4(\upsilon)}(t-u)\right] {}^{RL}\mathbb{D}_u^{1-\upsilon}\mathbf{T}(u)du, \\ {}^{CPC}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \frac{1}{\mathbb{M}_6(\upsilon)} \int_0^t exp\left[-\frac{\mathbb{M}_7(\upsilon)}{\mathbb{M}_6(\upsilon)}(t-u)\right] {}^{RL}\mathbb{D}_u^{1-\upsilon}\mathbf{S}(u)du. \end{cases} \tag{22}$$

and similarity,

$$\begin{cases} {}^{CPC}\mathbb{D}_t^{\upsilon}{}^{CPC}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) = \mathbf{S}(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \lim_{t\to 0} {}^{RL}\mathbb{I}_t^{\upsilon}\mathbf{S}(t), \\ {}^{CPC}\mathbb{D}_t^{\upsilon}{}^{CPC}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) = \mathbf{H}(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \lim_{t\to 0} {}^{RL}\mathbb{I}_t^{\upsilon}\mathbf{H}(t), \\ {}^{CPC}\mathbb{D}_t^{\upsilon}{}^{CPC}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) = \mathbf{T}(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \lim_{t\to 0} {}^{RL}\mathbb{I}_t^{\upsilon}\mathbf{T}(t), \\ {}^{CPC}\mathbb{D}_t^{\upsilon}{}^{CPC}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) = \mathbf{R}(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \lim_{t\to 0} {}^{RL}\mathbb{I}_t^{\upsilon}\mathbf{R}(t). \end{cases} \tag{23}$$

$$\begin{cases} {}^{CPC}\mathbb{I}_t^{\upsilon}{}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{S}(t) = \mathbf{S}(t) - exp\left[-\int_0^t \frac{\mathbb{M}_1(\upsilon,S)}{\mathbb{M}_0(\upsilon,S)}ds\right]\mathbf{S}(0), \\ {}^{CPC}\mathbb{I}_t^{\upsilon}{}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{H}(t) = \mathbf{H}(t) - exp\left[-\int_0^t \frac{\mathbb{M}_3(\upsilon,S)}{\mathbb{M}_2(\upsilon,S)}ds\right]\mathbf{H}(0), \\ {}^{CPC}\mathbb{I}_t^{\upsilon}{}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{T}(t) = \mathbf{T}(t) - exp\left[-\int_0^t \frac{\mathbb{M}_5(\upsilon,S)}{\mathbb{M}_4(\upsilon,S)}ds\right]\mathbf{T}(0), \\ {}^{CPC}\mathbb{I}_t^{\upsilon}{}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{R}(t) = \mathbf{R}(t) - exp\left[-\int_0^t \frac{\mathbb{M}_7(\upsilon,S)}{\mathbb{M}_6(\upsilon,S)}ds\right]\mathbf{R}(0). \end{cases} \tag{24}$$

Proof. The definitions (21) and (22) can be written as operational compositions ${}^{PC}\mathbb{I}_t^{\upsilon} = {}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}$ and ${}^{PC}\mathbb{I}_t^{\upsilon} = {}^{CPC}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}$, the known inversion relations for each component of each operator and the composition of operators lead to the inversion relations.

For Non-Drinkers Class:

$$\left({}^{PC}\mathbb{D}_t^{\upsilon} \bullet {}^{PC}\mathbb{I}_t^{\upsilon}\right)\mathbf{S}(t) = \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right) \bullet \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right)\mathbf{S}(t)$$

$$= \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right)\mathbf{S}(t)$$

$$= \mathbf{S}(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \lim_{t\to 0} {}^{RL}\mathbb{I}_t^{\upsilon}\mathbf{S}(t) \tag{25}$$

$$\left({}^{PC}\mathbb{I}_t^{\upsilon} \bullet {}^{PC}\mathbb{D}_t^{\upsilon}\right)\mathbf{S}(t) = \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right) \bullet \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right)\mathbf{S}(t)$$

$$= \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right)\mathbf{S}(t)$$

$$= \mathbf{S}(t) - exp\left(-\int_0^t \frac{\mathbb{M}_1(\upsilon,S)}{\mathbb{M}_0(\upsilon,S)}dS\right)\mathbf{S}(0) \tag{26}$$

For Heavy Drinkers Class:

$$\left({}^{PC}\mathbb{D}_t^{\upsilon} \bullet {}^{PC}\mathbb{I}_t^{\upsilon}\right)\mathbf{H}(t) = \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right) \bullet \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right)\mathbf{H}(t)$$

$$= \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right)\mathbf{H}(t)$$

$$= \mathbf{H}(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \lim_{t\to 0} {}^{RL}\mathbb{I}_t^{\upsilon}\mathbf{H}(t) \tag{27}$$

$$\left({}^{PC}\mathbb{I}_t^{\upsilon} \bullet {}^{PC}\mathbb{D}_t^{\upsilon}\right)\mathbf{H}(t) = \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right) \bullet \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right)\mathbf{H}(t)$$

$$= \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right)\mathbf{H}(t)$$

$$= \mathbf{H}(t) - exp\left(-\int_0^t \frac{\mathbb{M}_3(\upsilon,S)}{\mathbb{M}_2(\upsilon,S)}dS\right)\mathbf{H}(0) \tag{28}$$

For Drinkers in Treatment Class:

$$\left({}^{PC}\mathbb{D}_t^{\upsilon} \bullet {}^{PC}\mathbb{I}_t^{\upsilon}\right)\mathbf{T}(t) = \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right) \bullet \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right)\mathbf{T}(t)$$

$$= \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right)\mathbf{T}(t)$$

$$= \mathbf{T}(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \lim_{t\to 0} {}^{RL}\mathbb{I}_t^{\upsilon}\mathbf{T}(t) \tag{29}$$

$$\left({}^{PC}\mathbb{I}_t^{\upsilon} \bullet {}^{PC}\mathbb{D}_t^{\upsilon}\right)\mathbf{T}(t) = \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right) \bullet \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right)\mathbf{T}(t)$$

$$= \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right)\mathbf{T}(t)$$

$$= \mathbf{T}(t) - exp\left(-\int_0^t \frac{\mathbb{M}_5(\upsilon,S)}{\mathbb{M}_4(\upsilon,S)}dS\right)\mathbf{T}(0) \tag{30}$$

For Recovered Drinkers Class:

$$\left({}^{PC}\mathbb{D}_t^{\upsilon} \bullet {}^{PC}\mathbb{I}_t^{\upsilon}\right)\mathbf{R}(t) = \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right) \bullet \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right)\mathbf{R}(t)$$

$$= \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right)\mathbf{R}(t)$$

$$= \mathbf{R}(t) - \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \lim_{t\to 0} {}^{RL}\mathbb{I}_t^{\upsilon}\mathbf{R}(t) \tag{31}$$

$$\left({}^{PC}\mathbb{I}_t^{\upsilon} \bullet {}^{PC}\mathbb{D}_t^{\upsilon}\right)\mathbf{R}(t) = \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{RL}\mathbb{D}_t^{1-\upsilon}\right) \bullet \left({}^{RL}\mathbb{I}_t^{1-\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right)\mathbf{R}(t)$$

$$= \left({}^{P}\mathbb{I}_t^{\upsilon} \bullet {}^{P}\mathbb{D}_t^{\upsilon}\right)\mathbf{R}(t)$$

$$= \mathbf{R}(t) - exp\left(-\int_0^t \frac{\mathbb{M}_7(\upsilon,S)}{\mathbb{M}_6(\upsilon,S)}dS\right)\mathbf{R}(0) \tag{32}$$

as well as for the CPC operators. We have used the Riemann-Liouville differintegrals (inversion relations quote [18]), the $\mathbb{D}_a$ and $\mathbb{I}_a$ operators (inversion relations (18)-(20)), and composition formulations (13)-(16) for the PC and CPC derivatives. □

6. Eigenfunctions of the CPC Operator

In this part, we use the Laplace transform and Theorem **??** to solve a few differential equations using our new CPC derivative.

$$\begin{cases} {}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{S}(t) = b - \alpha\mathbf{S}\mathbf{H} - \mu\mathbf{S} + \eta\mathbf{R}, \\ {}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{H}(t) = \alpha\mathbf{S}\mathbf{H} - (\mu + \delta_1 + \phi)\mathbf{H}, \\ {}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{T}(t) = \phi\mathbf{H} - (\mu + \delta_2 + \gamma)\mathbf{T}, \\ {}^{CPC}\mathbb{D}_t^{\upsilon}\mathbf{R}(t) = \gamma\mathbf{T} - (\mu + \eta)\mathbf{R}. \end{cases} \tag{33}$$

Using the starting condition and the Laplace transform on both sides, and after simplification, we have

$$\begin{cases} \widehat{\mathbf{S}}(S) = \frac{\mathbb{M}_0(\upsilon)S^{\upsilon-1}\mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}}{\left[\frac{\mathbb{M}_1(\upsilon)}{S}+\mathbb{M}_0(\upsilon)\right]S^{\upsilon}}, \\ \widehat{\mathbf{H}}(S) = \frac{\mathbb{M}_2(\upsilon)S^{\upsilon-1}\mathbf{S}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}}{\left[\frac{\mathbb{M}_3(\upsilon)}{S}+\mathbb{M}_2(\upsilon)\right]S^{\upsilon}}, \\ \widehat{\mathbf{T}}(S) = \frac{\mathbb{M}_4(\upsilon)S^{\upsilon-1}\mathbf{S}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}}{\left[\frac{\mathbb{M}_5(\upsilon)}{S}+\mathbb{M}_4(\upsilon)\right]S^{\upsilon}}, \\ \widehat{\mathbf{R}}(S) = \frac{\mathbb{M}_6(\upsilon)S^{\upsilon-1}\mathbf{S}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}}{\left[\frac{\mathbb{M}_7(\upsilon)}{S}+\mathbb{M}_6(\upsilon)\right]S^{\upsilon}}. \end{cases}$$

$$\begin{cases} \widehat{\mathbf{S}}(S) = \frac{S^{-1}\mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}}{\left[\frac{\mathbb{M}_1(\upsilon)}{S\mathbb{M}_0(\upsilon)}+1\right]}, \\ \widehat{\mathbf{H}}(S) = \frac{S^{-1}\mathbf{H}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}}{\left[\frac{\mathbb{M}_3(\upsilon)}{S\mathbb{M}_2(\upsilon)}+1\right]}, \\ \widehat{\mathbf{T}}(S) = \frac{S^{-1}\mathbf{T}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}}{\left[\frac{\mathbb{M}_5(\upsilon)}{S\mathbb{M}_4(\upsilon)}+1\right]}, \\ \widehat{\mathbf{R}}(S) = \frac{S^{-1}\mathbf{R}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}}{\left[\frac{\mathbb{M}_7(\upsilon)}{S\mathbb{M}_6(\upsilon)}+1\right]}. \end{cases}$$

$$\begin{cases} \widehat{\mathbf{S}}(S) = S^{-1}\mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\left[1+\frac{\mathbb{M}_1(\upsilon)S^{-1}}{\mathbb{M}_0(\upsilon)}\right]^{-1}, \\ \widehat{\mathbf{H}}(S) = S^{-1}\mathbf{H}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\left[1+\frac{\mathbb{M}_3(\upsilon)S^{-1}}{\mathbb{M}_2(\upsilon)}\right]^{-1}, \\ \widehat{\mathbf{T}}(S) = S^{-1}\mathbf{T}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\left[1+\frac{\mathbb{M}_5(\upsilon)S^{-1}}{\mathbb{M}_4(\upsilon)}\right]^{-1}, \\ \widehat{\mathbf{R}}(S) = S^{-1}\mathbf{R}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\left[1+\frac{\mathbb{M}_7(\upsilon)S^{-1}}{\mathbb{M}_6(\upsilon)}\right]^{-1}. \end{cases}$$

$$\begin{cases} \widehat{\mathbf{S}}(S) = S^{-1}\mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\left[-\frac{\mathbb{M}_1(\upsilon)S^{-1}}{\mathbb{M}_0(\upsilon)}\right]^{n}, \\ \widehat{\mathbf{H}}(S) = S^{-1}\mathbf{H}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\left[-\frac{\mathbb{M}_3(\upsilon)S^{-1}}{\mathbb{M}_2(\upsilon)}\right]^{n}, \\ \widehat{\mathbf{T}}(S) = S^{-1}\mathbf{T}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\left[-\frac{\mathbb{M}_5(\upsilon)S^{-1}}{\mathbb{M}_4(\upsilon)}\right]^{n}, \\ \widehat{\mathbf{R}}(S) = S^{-1}\mathbf{R}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\left[-\frac{\mathbb{M}_7(\upsilon)S^{-1}}{\mathbb{M}_6(\upsilon)}\right]^{n}. \end{cases}$$

$$\begin{cases} \widehat{\mathbf{S}}(S) = S^{-1}\mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\frac{1}{\mathbb{M}_0(\upsilon)^n}\sum_{k=0}^{n}\binom{n}{k}\left[-\mathbb{M}_1(\upsilon)S^{-1}\right]^{k}, \\ \widehat{\mathbf{H}}(S) = S^{-1}\mathbf{H}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\frac{1}{\mathbb{M}_2(\upsilon)^n}\sum_{k=0}^{n}\binom{n}{k}\left[-\mathbb{M}_3(\upsilon)S^{-1}\right]^{k}, \\ \widehat{\mathbf{T}}(S) = S^{-1}\mathbf{T}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\frac{1}{\mathbb{M}_4(\upsilon)^n}\sum_{k=0}^{n}\binom{n}{k}\left[-\mathbb{M}_5(\upsilon)S^{-1}\right]^{k}, \\ \widehat{\mathbf{R}}(S) = S^{-1}\mathbf{R}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\frac{1}{\mathbb{M}_6(\upsilon)^n}\sum_{k=0}^{n}\binom{n}{k}\left[-\mathbb{M}_7(\upsilon)S^{-1}\right]^{k}. \end{cases}$$

$$\begin{cases} \widehat{\mathbf{S}}(S) = S^{-1}\mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_1(\upsilon))^k}{\mathbb{M}_0(\upsilon)^n}\binom{n}{k}S^{-k}, \\ \widehat{\mathbf{H}}(S) = S^{-1}\mathbf{H}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_3(\upsilon))^k}{\mathbb{M}_2(\upsilon)^n}\binom{n}{k}S^{-k}, \\ \widehat{\mathbf{T}}(S) = S^{-1}\mathbf{T}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_5(\upsilon))^k}{\mathbb{M}_4(\upsilon)^n}\binom{n}{k}S^{-k}, \\ \widehat{\mathbf{R}}(S) = S^{-1}\mathbf{R}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_7(\upsilon))^k}{\mathbb{M}_6(\upsilon)^n}\binom{n}{k}S^{-k}. \end{cases}$$

$$\begin{cases} \widehat{\mathbf{S}}(S) = \mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_1(\upsilon))^k}{\mathbb{M}_0(\upsilon)^n}\binom{n}{k}S^{-k-1}, \\ \widehat{\mathbf{H}}(S) = \mathbf{H}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_3(\upsilon))^k}{\mathbb{M}_2(\upsilon)^n}\binom{n}{k}S^{-k-1}, \\ \widehat{\mathbf{T}}(S) = \mathbf{T}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_5(\upsilon))^k}{\mathbb{M}_4(\upsilon)^n}\binom{n}{k}S^{-k-1}, \\ \widehat{\mathbf{R}}(S) = \mathbf{R}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_7(\upsilon))^k}{\mathbb{M}_6(\upsilon)^n}\binom{n}{k}S^{-k-1}. \end{cases}$$

By term-by-term using the inverse Laplace transform, we discover

$$\begin{cases} \mathbf{S}(S) = \mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_1(\upsilon))^k}{\mathbb{M}_0(\upsilon)^n}\binom{n}{k}\frac{t^k}{\Gamma(k+1)}, \\ \mathbf{H}(S) = \mathbf{H}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_3(\upsilon))^k}{\mathbb{M}_2(\upsilon)^n}\binom{n}{k}\frac{t^k}{\Gamma(k+1)}, \\ \mathbf{T}(S) = \mathbf{T}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_5(\upsilon))^k}{\mathbb{M}_4(\upsilon)^n}\binom{n}{k}\frac{t^k}{\Gamma(k+1)}, \\ \mathbf{R}(S) = \mathbf{R}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_7(\upsilon))^k}{\mathbb{M}_6(\upsilon)^n}\binom{n}{k}\frac{t^k}{\Gamma(k+1)}. \end{cases}$$

$$\begin{cases} \mathbf{S}(S) = \mathbf{S}(0)G_1\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_1(\upsilon))^k}{\mathbb{M}_0(\upsilon)^n}\left(\frac{n!}{k!(n-k)!}\right)\frac{t^k}{\Gamma(k+1)}, \\ \mathbf{H}(S) = \mathbf{H}(0)G_2\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_3(\upsilon))^k}{\mathbb{M}_2(\upsilon)^n}\left(\frac{n!}{k!(n-k)!}\right)\frac{t^k}{\Gamma(k+1)}, \\ \mathbf{T}(S) = \mathbf{T}(0)G_3\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_5(\upsilon))^k}{\mathbb{M}_4(\upsilon)^n}\left(\frac{n!}{k!(n-k)!}\right)\frac{t^k}{\Gamma(k+1)}, \\ \mathbf{R}(S) = \mathbf{R}(0)G_4\{\mathbf{S},\mathbf{H},\mathbf{T},\mathbf{R}\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-\mathbb{M}_7(\upsilon))^k}{\mathbb{M}_6(\upsilon)^n}\left(\frac{n!}{k!(n-k)!}\right)\frac{t^k}{\Gamma(k+1)}. \end{cases}$$

The recently defined bivariate Mittag-Leffler function [20] can be used to represent this series.

$$\begin{cases} \mathbf{S}(S) = \mathrm{E}^1_{\upsilon,1,1}\left(\frac{t^\upsilon}{\mathbb{M}_0(\upsilon)}, \frac{-t\mathbb{M}_1(\upsilon)}{\mathbb{M}_0(\upsilon)}\right), \\ \mathbf{H}(S) = \mathrm{E}^1_{\upsilon,1,1}\left(\frac{t^\upsilon}{\mathbb{M}_2(\upsilon)}, \frac{-t\mathbb{M}_3(\upsilon)}{\mathbb{M}_2(\upsilon)}\right), \\ \mathbf{T}(S) = \mathrm{E}^1_{\upsilon,1,1}\left(\frac{t^\upsilon}{\mathbb{M}_4(\upsilon)}, \frac{-t\mathbb{M}_5(\upsilon)}{\mathbb{M}_4(\upsilon)}\right), \\ \mathbf{R}(S) = \mathrm{E}^1_{\upsilon,1,1}\left(\frac{t^\upsilon}{\mathbb{M}_6(\upsilon)}, \frac{-t\mathbb{M}_7(\upsilon)}{\mathbb{M}_6(\upsilon)}\right). \end{cases} \tag{34}$$

7. Conclusion

In this study, we proposed two novel fractional derivatives to the drinking epidemic that are closely connected to one another and may be described as a combination (or hybridization) of separate existing fractional operators. They were first produced by beginning with the Caputo fractional derivative and substituting a proportional derivative for the simple derivative. One of them is a linear combination of a Riemann-Liouville integral with a Caputo derivative, which we referred to as CPC rather than PC. In order to solve the drinking epidemic mathematical model using the CPC derivative, we highlighted the close relationship between fractional calculus and Mittag-Leffler functions. The answer, which was arrived at using Laplace transform techniques, may be expressed in terms of a brand-new bivariate Mittag-Leffler function, which was just created and is already finding many uses. We were able to demonstrate the uniqueness and Ulam-Hyres stability of solutions to a particular class of fractional starting value issue involving the Hilfer proportional fractional derivative using certain well-known theorems from the fixed point theory. In contrast to the single formula for the inverse of the PC derivative, we computed the Laplace transform and discovered two alternative equations for its inverse operator also treated for model. We noted that drinking, because of uncertainty related to the pandemic, shows deficiencies in ordinal derivatives and their associated integral operators, which we used to quantitatively represent this sickness. This is because fractional derivatives and integrals make it possible to describe the memory and hereditary features that are built into different materials and processes. As a result, it is becoming more important to understand and apply fractional order differential equations.Study is also helpful to overcome the bad impact of drinking on health and economic which is major cause society disasters.

Funding: None

Conflict of interest: None

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