Integral Identity Method to Fluid Flow through Cracked Porous Media with Different Wetting Abilities

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ABSTRACT: It is a notable actual peculiarities when a permeable media is totally immersed with a non-wetting liquid. For instance, water is brought into contact. The last option will more often than not precipitously stream into the medium, dislodging the non-wetting liquid. In this paper, we utilize indispensable personalities with intersecting hyper-mathematical series to address the progression of two immiscible fluids in a broke permeable medium. The methodology adopted for the solution is followed by transform of non-linear differential system into an ordinary differential equation. Subsequently obtained equation is convert into diffusion equation by applying similarity variable by standard transformation and further transfer into the confluent hyper geometric equation. The acquired arrangement as far as intersecting hyper mathematical series give an articulation for wetting stage immersion. The outcomes exhibit the straightforward examination to acquire a scientific arrangement of the non-direct differential condition of imbibitions peculiarity under extraordinary condition in a broke permeable media wherein the water infiltrating the crease along the broke is sucked into the squares of rock under the activity of hairlike powers and how much water entering the square in the rudimentary volume.

KEYWORDS: Capillary Forces, Confluent Hyper Geometric Series, Cracked Porous Media, Diffusion Equation, Imbibitions Phenomenon, Non-linear Differential Equation.

I. INTRODUCTION

When a porous medium is completely saturated with a non-wetting fluid, such as water, it is a well-known physical fact that. The later will tend to flow spontaneously into the medium displacing the non-wetting fluid. Such phenomena named as imbibitions phenomena and it has been discussed by BROWSCOMBE and DYES (1952)[1]–[3], From an analytical standpoint, GRAHAM has looked at two unique oil-water displacement processes. We've looked at it from an analytical standpoint in this chapter [4], [5]. The non-linear differential system is transformed into an ordinary differential equation and then convert it into diffusion equation by applying similarity variable which is further by standard transformation; transformed into confluent hyper geometric equation and its solution is obtain in terms of confluent hyper geometric series which gives an appearance for wetting phase saturations.

A. Declaration of the problems

Consider a semi-limitless length round and hollow piece of oil-immersed cracked permeable material. Which is verged on three sides by impermeable surfaces and is open and presented to a close by water arrangement. The peculiarity of straight counter current imbibition is brought about by this arrangement. A few standard outcomes for the connection between relative penetrability and stage immersion, impregnation work, oil-water consistency proportion, and hairlike tension reliance on stage immersion are accommodated clearness.

The fundamental premium of the current examination is to acquire a scientific arrangement of the non-straight differential condition of imbibitions under extraordinary state of our problem.

B. Flow in cracked media

In a broke permeable media water infiltrating the crease along the broke is sucked into the squares of rock under the activity of hairlike powers and how much water entering the square in the rudimentary volume of crease is assigned as the impregnation work Ø(t),where t indicates the time [6]–[8].

Consider the equilibrium of water sucked into the squares of rock per unit time and emploing the consequence of MATTAX and KYTE [9], [10], VAZIROV, for the insightful worth of Ø we might compose

$$\emptyset|T-\tau(u)| = \left[D/(T-Rx^2)^{\frac{3}{2}}\right] \tag{1}$$

$$T = t\left(\frac{\delta cos\theta\, S^2\sqrt{\frac{k}{m_B}}}{V_0}\right)$$

$$D = \frac{A}{2}m_B q_k\left(\delta cos\theta + S^2\sqrt{\frac{k}{m_b}}\right)$$

$$R = \frac{1}{L_M^2}\left(\frac{\pi}{4q}S^2 cos\theta m_B g_k\sqrt{\frac{k}{m_B}}\right)$$

Where m_B= porosity of the blocks
g_k = Saturation of the block with water $t=t_k$

S = Mean exact surface area of the blocks
θ_n = wetting angle
V_0 = oil viscosity
V_w = the viscosity of water
K = the permeability of the crack system
A = constant coefficient
L_m =Mean block size
q = the consistent pace of conveyance of water per unit surface region opposite to the heading of water

It may be mentioned that we consider "q" as the average rate of flow of water across the imbibition face and assumed it to be constant in the present discussion.

II. DISCUSSION

Formulation of the problem:

DARCY's regulation gives the drainage speed of water (V_w) and oil (V_o) as.

$$V_w = -\frac{K_w}{V_w} k \frac{\partial P_w}{\partial x} \quad (2)$$

$$V_o = -\frac{K_o}{V_o} k \frac{\partial P_o}{\partial x} \quad (3)$$

Since $V_w = -V_o$ for the imbibition phenomenon, therefore. From equation $(2) as\ well\ as\ (3)$ we may write,

$$\frac{K_w}{V_w}\frac{\partial P_w}{\partial x} + \frac{K_o}{V_0}\frac{\partial P_o}{\partial x} = 0 \quad (4)$$

Now the pressure discontinuity between the flowing phase[11], [12]
yeild the defination of capillary pressure as

$$P_c = P_o - P_w \quad (5)$$

Combining $(4) and$ (5), We get,

$$\left(\frac{K_w}{V_w} + \frac{K_o}{V_o}\right)\frac{\partial P_w}{\partial x} + \frac{K_o}{V_o}\frac{\partial P_c}{\partial x} = 0 \quad (6)$$

Substituting the value of $\frac{\partial P_w}{\partial x}$ from (6) into (2), we get,

$$V_w = \frac{K\frac{K_w}{V_w}\frac{K_o}{V_o}\frac{\partial P_c}{\partial x}}{\frac{K_w}{V_w} + \frac{K_o}{V_o}} \quad (7)$$

Following RIJIK [4], the condition of progression for water might be composed as,

$$P\frac{\partial S_w}{\partial T} + \frac{\partial V_w}{\partial X} + \emptyset[T - \tau(u)] = 0 \quad (8)$$

Where $\emptyset|[T - \tau(u)]|$ is the impregnation functions, substituting the value of V_w and $\emptyset[T - \tau(u)]$ from equation $(7)\ and\ (1)$ into (8), We get,

$$\epsilon_p \frac{\partial S_w}{\partial T} + \frac{\partial}{\partial X}\left[K\frac{K_w K_o}{\vartheta_o K_w + K_o\vartheta_w}\right]\frac{\partial P_c}{\partial S_w}.D(T - R_x^2)^{-\frac{3}{2}} = 0 \quad (9)$$

Where $\varepsilon = \frac{\delta cos\theta S^2\sqrt{\frac{K}{m_B}}}{\vartheta_o}$

Equation (9) is non-straight differential condition which portrays the direct counter current imbibition peculiarity in a broke round and hollow framework with the limit condition.

$$S_w(0,T) = S_{w_0}: \frac{\partial S_w(L,T)}{\partial X} = 0$$

L is the half-length of a cylinder of oil-saturated cracked porous material. Which is surrounded on three sides by an impermeable surface and is open and exposed to a neighboring water formation. T is the above-mentioned function.

Method of Integral Identities:

It is well known that P_c is decreasing function of S_w (MUSKAT, 1949). Therefore, we may write.

$$P_c = -\beta S_w \quad (Mehta\ [12]) \quad (10)$$

Where negative sign indicates the direction of flow.

Also for definiteness. We assume that,

$$\frac{K_w K_o}{\vartheta_o K_w + K_o\vartheta_w} = \frac{K_o}{\vartheta_o} \quad (11)$$

Using$(10)\ and\ (11)\ into\ (9)$, it reduce to,

$$\varepsilon_p\frac{\partial S_w}{\partial T} + \frac{\partial}{\partial X}\left[\left(K\frac{K_w K_o}{\vartheta_o K_w + K_o\vartheta_w}\right)\left(\frac{dP_c}{dS_w}\right)\left(\frac{\partial S_w}{\partial X}\right)\right] = -D(T - R_x^2)^{-\frac{3}{2}}$$

Where $\varepsilon = \frac{\delta cos\theta S^2\sqrt{\frac{K}{m_B}}}{\vartheta_o}$

$$\varepsilon_p\frac{\partial S_w}{\partial T} + \frac{\partial}{\partial X}\left[\left(K\frac{K_w K_o}{\vartheta_o K_w + K_o\vartheta_w}\right)\left(\frac{dP_c}{dS_w}\right)\left(\frac{\partial S_w}{\partial X}\right)\right] = -D(T - R_x^2)^{-\frac{3}{2}}$$

$$\varepsilon_p\frac{\partial S_w}{\partial T} + \frac{\partial}{\partial X}\left[K\frac{K_o}{\vartheta_o}\frac{dP_c}{dS_w}\frac{\partial S_w}{\partial X}\right] = -D(T - R_x^2)^{-\frac{3}{2}}$$

$$\varepsilon_p\frac{\partial S_w}{\partial T} + K\frac{K_o}{\vartheta_o}\frac{dP_c}{dS_w}\frac{\partial^2 S_w}{\partial X^2} = -D(T - R_x^2)^{-\frac{3}{2}}$$

$$\frac{\partial S_w}{\partial T} + \frac{\beta_1}{\varepsilon_p}\frac{\partial^2 S_w}{\partial X^2} = -\frac{D}{\varepsilon_p}(T - R_x^2)^{-\frac{3}{2}} \quad (12)$$

Where $\beta_1 = -K\frac{K_o}{\vartheta_o}\frac{dP_c}{dS_w}$

Here P_c is linear function of S_w then $\frac{dP_c}{dS_w}$ is constant. So that β_1 is constant [13]

Applying the similarity variable viz,

$$S_w = \left[\frac{DF(Z)}{\lambda_2\sqrt{\lambda_1 T}}\right],\ Z = \frac{X}{2\sqrt{\lambda_1 T}} \quad (13)$$

Where $\lambda_1 = \frac{\beta_1}{\varepsilon_p}\ and\ \lambda_2 = -\frac{D}{\varepsilon_p}$

The equation (12) is transformed into ordinary differential equation viz.

$$F''(Z) + 2ZF'(Z) + 2F(Z) = \mu(1 - 4R\lambda_1 Z^2)^{-3/2} \quad (14)$$

$$F(0) = S_w,\ F'(L) = 0$$

Where $\mu = \frac{4D\sqrt{\beta_1}}{(\varepsilon_p)^3}$ and $\lambda_1 = \frac{\beta_1}{\varepsilon_p}$ are the small parameters.

Let us change the above ordinary differential equation into diffusion equation.

$$-\frac{d}{dx}\left[P\frac{d\phi}{dx}\right] + q\phi = f$$

$$-P\frac{d^2\phi}{dx^2} - \frac{dP}{dx}\frac{d\phi}{dx} + q\phi = f$$

$$\frac{d^2\phi}{dx^2} + \frac{1}{P}\frac{dP}{dx}\frac{d\phi}{dx} + \frac{q\phi}{P} = -\frac{f}{P} \quad (15)$$

Now our ordinary differential equation is,

$$\frac{d^2F(x)}{dx^2} + 2\frac{dF(x)}{dx} + 2F(x) = \mu(1 - 4R\lambda_1 Z^2)^{-3/2} \quad (16)$$

$F(0) = S_w\,, F'(L) = 0$

Equation (15) and (16), we obtain

If two function are equate then we should equate just like it. If all the multiplier are same than we should equate and take it as any constant K.

$$\frac{1}{P}\frac{dP}{dx} = 2cx \qquad Also - \frac{q}{P} = 2c$$

$$P = e^{cx^2} \qquad q = -2ce^{cx^2}$$

The diffusion equation becomes,

$$\frac{d}{dx}\left(e^{cx^2}\frac{dF}{dx}\right) - 2ce^{cx^2}F(X) = -\mu e^{-cx^2}(1-4R\lambda_1 x^2)^{-3/2}$$

$$\frac{d}{dx}\left(e^{cx^2}\frac{dF}{dx}\right) + 2ce^{cx^2}F(X) = \mu e^{-cx^2}(1+4R\lambda_1 x^2)^{-\frac{3}{2}} \qquad (17)$$

With boundary condition $F(0) = 0\,, F(1) = 0$

Where $P = P(X) = e^{cx^2}$ is diffusion

$$Q = 2ce^{cx^2}$$

$$F = f(x) = e^{-cx^2}\mu(1+4R\lambda_1 x^2)^{-\frac{3}{2}}$$

$F = f(x) = e^{-cx^2}\mu(1+4R\lambda_1 x^2)^{-\frac{3}{2}}$ is the source of diffusion

Let us suppose that the functions are piecewise continuous with discontinuities of the first kind. We wish to find continuous solution of (14) Which has a differential 'Flow '.

$J = e^{cx^2}\frac{dF}{dx}$

Which is satisfies the boundary condition.

$F(0) = 0$ and $F(1) = 0 \qquad (18)$

Let us choose two system of net points over the range [0,1] of variable x.

Now participant (i) the basic system $\{x_k\}_{k=0}^{n}$ and (ii) the auxiliary system $\left\{x_{k+\frac{1}{2}}\right\}_{k=0}^{n}$

The point from these two systems are mutually alternative in succession.

$i.e.\ x_k < x_{k+\frac{1}{2}} < x_{k+1}$ and $x_0 = 1, x_n = 1$

We will assume that. $x_{k+\frac{1}{2}} = \left(\frac{x_k + x_{k+1}}{2}\right)$

Integrating (17) with respect to x from $x_{k-\frac{1}{2}}$ to $x_{k+\frac{1}{2}}$. As result, we obtain that the equilibrium relation.

$$-\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\frac{d}{dx}\left(e^{cx^2}\frac{dF}{dx}\right)dx + \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} 2ce^{cx^2}F(x)dx = -\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\mu e^{-cx^2}(1+4R\lambda_1 x^2)^{-\frac{3}{2}}dx$$

$$\left[-e^{cx^2}\frac{dF}{dx}\right]_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} + \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} 2ce^{cx^2}F(x)dx = -\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\mu e^{-cx^2}(1+4R\lambda_1 x^2)^{-\frac{3}{2}}dx$$

$$-\left[e^{c\left(x_{k+\frac{1}{2}}\right)^2}\frac{dF\left(x_{k+\frac{1}{2}}\right)}{dx} - e^{c\left(x_{k-\frac{1}{2}}\right)^2}\frac{dF\left(x_{k-\frac{1}{2}}\right)}{dx}\right] + \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x) - \mu e^{-cx^2}(1+4R\lambda_1 x^2)^{-\frac{3}{2}}\right)dx = 0$$

$$-J\left(x_{k+\frac{1}{2}}\right) + J\left(x_{k-\frac{1}{2}}\right) + \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x) - \mu e^{-cx^2}(1+4R\lambda_1 x^2)^{-\frac{3}{2}}\right)dx = 0 \quad (19)$$

Where $J\left(x_{k+\frac{1}{2}}\right) = J\left(x_{k-\frac{1}{2}}\right)$

In order to find $J_{k\pm\frac{1}{2}}$, process is as follows,

Integrating (18) with respect to x from $x_{k-\frac{1}{2}}$ to x

$$-\int_{x_{k-\frac{1}{2}}}^{x}\frac{d}{dx}\left(e^{cx^2}\frac{dF(x)}{dx}\right)dx + \int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi = \int_{x_{k-\frac{1}{2}}}^{x}\mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}d\xi$$

$$-\left(e^{cx^2}\frac{dF(x)}{dx}\right)_{x_{k-\frac{1}{2}}}^{x} + \int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi = \int_{x_{k-\frac{1}{2}}}^{x}\mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}d\xi$$

$$-\left(e^{cx^2}\frac{dF(x)}{dx}\right) + \left(e^{cx^2_{k-\frac{1}{2}}}\frac{dF\left(x_{k-\frac{1}{2}}\right)}{dx}\right) + \int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi = \int_{x_{k-\frac{1}{2}}}^{x}\mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}d\xi$$

$$-\left(e^{cx^2}\frac{dF(x)}{dx}\right) + \left(e^{cx^2_{k-\frac{1}{2}}}\frac{dF\left(x_{k-\frac{1}{2}}\right)}{dx}\right) + \int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi - \int_{x_{k-\frac{1}{2}}}^{x}\mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}d\xi = 0$$

$$e^{cx^2}\frac{dF(x)}{dx} = J_{k-\frac{1}{2}} + \int_{x_{k-\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}F(\xi) - \mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi$$

$$\frac{dF(x)}{dx} = e^{-cx^2}J_{k-\frac{1}{2}} + e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}F(\xi) - \mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi \qquad (20)$$

Integrating (20) with respect to $x\ from\ x_{k-1}\ to\ x_k$

$$\int_{x_{k-1}}^{x_k}\frac{dF(x)}{dx}dx = \int_{x_{k-1}}^{x_k} e^{-c\xi^2}J_{k-\frac{1}{2}}dx + \int_{x_{k-1}}^{x_k} e^{-c\xi^2}\int_{x_{k-\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}F(\xi) - \mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx$$

$$[F(x)]_{x_{k-1}}^{x_k} = J_{k-\frac{1}{2}}\int_{x_{k-1}}^{x_k} e^{-cx^2}dx + \int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}F(\xi) - \mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx$$

$$F(x_k) - F(x_{k-1}) = J_{k-\frac{1}{2}}\int_{x_{k-1}}^{x_k} e^{-cx^2}dx + \int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}F(\xi) - \mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx$$

Divide the above equation by $\int_{x_{k-1}}^{x_k} e^{-cx^2}dx$

$$\frac{F(x_k) - F(x_{k-1})}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} = J_{k-\frac{1}{2}} + \frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}F(\xi) - \mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx}$$

$$J_{k-\frac{1}{2}} = \frac{F(x_k) - F(x_{k-1})}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}F(\xi) - \mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} \qquad (22)$$

A similar expression is obtained for the $J_{k+\frac{1}{2}}$ by taking $(k+1)$ rather than k in (22). In this way we have managed to express the flows $J_{k\pm\frac{1}{2}}$ by means of known functions of the problem. The relation (22) is exact.

$$J_{k+\frac{1}{2}} = \frac{F(x_{k+1}) - F(x_k)}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}F(\xi) - \mu e^{-c\xi^2}(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx}$$

A substitution of (22) and the corresponding $J_{k+\frac{1}{2}}$ $J_{k+½}$ into (19), namely,

$$-J_{k-\frac{1}{2}} + J_{k+\frac{1}{2}} + \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x) - \mu e^{-cx^2}(1+4R\lambda_1x^2)^{-\frac{3}{2}}\right)dx = 0$$

$$\therefore \left\{\frac{F(x_{k+1}) - F(x_k)}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}f(\xi) - e^{-c\xi^2}\mu(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx}\right\}$$
$$+\left\{\frac{F(x_k) - F(x_{k-1})}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}f(\xi) - e^{-c\xi^2}\mu(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx}\right\}$$
$$+ \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x) - e^{-cx^2}\mu(1+4R\lambda_1x^2)^{-\frac{3}{2}}\right)dx =$$

$$-\frac{F(x_{k+1}) - F(x_k)}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} + \frac{F(x_k) - F(x_{k-1})}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} + \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x) - e^{-cx^2}\mu(1+4R\lambda_1x^2)^{-\frac{3}{2}}\right)dx$$
$$= -\frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x}\left(2ce^{c\xi^2}f(\xi) - e^{-c\xi^2}\mu(1+4R\lambda_1\xi^2)^{-\frac{3}{2}}\right)d\xi\, dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} \qquad (21)$$

Equation (23) is the basic identity to be used for obtaining the finite differential equation.

Now we defined the operation A on the domain $\emptyset$ of the solution (17) as follow where F is the Hilbert space $L_2(D)$. Where D *is* $n-dimensional$ Euclidean space E_n. We denote by $L_2(D)$ the Hilbert Space of all real measurable square integrable functions $\int_D f^2(x)dx < \infty$ with the inner product $(f,g) = \int_D f(x)g(x)dx$

The norm of the function $f \in L_2(D)$ *is defined by* $\|f\| = (f,f)^{\frac{1}{2}}$

$$(AF)_k = -\frac{1}{\Delta x_k}\left[\frac{F(x_{k+1})-F(x_k)}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{F(x_k)-F(x_{k-1})}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x)dx\right) - \frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} + \frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx}\right]$$

$$(f)_k = -\frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f dx - \frac{1}{\Delta x_k}\left[\frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x} f d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} f d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx}\right]$$

$$(AF)_k = -\frac{1}{\Delta x_k}\left[\frac{F(x_{k+1})-F(x_k)}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{F(x_k)-F(x_{k-1})}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x)dx\right) - \frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} + \frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx}\right] \quad (24)$$

Also consider the vector f with the component,

$$(f)_k = -\frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f dx - \frac{1}{\Delta x_k}\left[\frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x} f d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}\,dx} - \frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} f d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}\,dx}\right] \quad (25)$$

Where $(f)_k$ is a vector and f is the source of diffusion substance and

$$f(x) = e^{-cx^2}\mu(1+4R\lambda_1 x^2)^{-\frac{3}{2}}$$

Also $\Delta x_k = x_{k+\frac{1}{2}} - x_{k-\frac{1}{2}}$ $where\ k = 1,2,3,\dots,(n-1)$.

For simplicity we will assume that the solution of (17) are chosen from the class ϕ each function of which has certain smoothness properties and satisfied the boundary condition $F(x) = 0$.

Using a more compact notation (23) for $k = 1,2,3,\dots(n-1)$ can be written as

$$AF = f \quad (26)$$

Consider the further various approximation of equation(26). Let us introduce the Euclidean Form,

$$\|F\|_{\phi_h}^2 = \sum_{k=1}^{n-1}\left(F_k^h\right)^2 \Delta x_k \quad (27)$$

Where $\phi_h$the space of net functions from is $F_h = (F_1^h, F_2^h, \dots, F_{n-1}^h)$ defined at points $x_1, x_2, \dots, x_{n-1}$. consider the following approximation,

$$A^h F^h = f^h \quad (28)$$

Where $(A^hF^h)_k = -\frac{1}{\Delta x_k}\left[\frac{F^h(x_{k+1})-F^h(x_k)}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{F^h(x_k)-F^h(x_{k-1})}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - F_k^h\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}dx\right)\right]$ (29)

Now we derive the value of $\xi^h, \eta^h, and\ \phi^h$

$\left((AF)_h - A^h(F_h)\right) =$

$$= -\frac{1}{\Delta x_k}\left[\frac{F_{k+1}-F_k}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{F_k - F_{k-1}}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x)dx\right) - \frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} + \frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx}\right] + \frac{1}{\Delta x_k}\left[\frac{F_{k+1}^h - F_k^h}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{F_k^h - F_{k-1}^h}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - F_k^h\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}dx\right)\right]$$

$$= -\frac{1}{\Delta x_k}\frac{F_{k+1}-F_k}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} + \frac{1}{\Delta x_k}\frac{F_k - F_{k-1}}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} + \frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}F(x)dx\right) + \frac{1}{\Delta x_k}\frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{1}{\Delta x_k}\frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} + \frac{1}{\Delta x_k}\frac{F_{k+1}^h - F_k^h}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{1}{\Delta x_k}\frac{F_k^h - F_{k-1}^h}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \frac{1}{\Delta x_k}F_k^h\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}dx\right)$$

$$= \frac{1}{\Delta x_k}\left[\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}\right)F(x)dx - F_k^h\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}\right)dx\right]$$

$$-\frac{1}{\Delta x_k}\left[\frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx}\right]$$

$$\xi^h = \frac{1}{\Delta x_k}\left[\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}\right)F(x)dx - F_k^h\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}\right)dx\right] \quad (1)$$

$$\eta^h = -\frac{1}{\Delta x_k}\left[\frac{\int_{x_{k-1}}^{x_k} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x} 2ce^{c\xi^2}F(\xi)d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx}\right] \quad (2)$$

Now

$$\theta^h = [(f)_h - f^h]$$

$$= \frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f dx - \frac{1}{\Delta x_k}\left[\frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x_k} f d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{\int_{x_{k-1}}^{x} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} f d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx}\right] - \frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f dx$$

$$= \frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f(x)dx - \frac{1}{\Delta x_k}\left[\int_{x_{k+\frac{1}{2}}}^{x} f(x)dx - \int_{x_{k-\frac{1}{2}}}^{x} f(x)dx\right] - \frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f(x)dx$$

$$= \frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f(x)dx - \frac{1}{\Delta x_k}\frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x_k} f d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} + \frac{1}{\Delta x_k}\frac{\int_{x_{k-1}}^{x} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} f d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} - \frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f dx$$

$$\theta^h = -\frac{1}{\Delta x_k}\left[\frac{\int_{x_k}^{x_{k+1}} e^{-cx^2}\int_{x_{k+\frac{1}{2}}}^{x_k} f d\xi dx}{\int_{x_k}^{x_{k+1}} e^{-cx^2}dx} - \frac{\int_{x_{k-1}}^{x} e^{-cx^2}\int_{x_{k-\frac{1}{2}}}^{x} f d\xi dx}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx}\right]$$

Where $(F^h)_k = -\frac{1}{\Delta x_k}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f dx$ $for\ k = 1,2,\dots,n-1\ and\ F_0^h = F_n^h = 0$

Using the triangular inequality, we have,

$$\|(AF)_h - A^h(F_h)\|_{\emptyset_h} \le \|\xi^h\|_{\phi_h} + \|\eta^h\|_{\phi_h}$$

And $\|(f)_h - f^h\| = \|\theta^h\|_{\phi_h}$ (30)

For any continuous function $u\ on\ [0,1] we\ take$ symbol $(u)_h$ to denote the $n-1$ dimensional vector from ϕ_h with the components $u(x_k)$.

Let us estimate the norms $\|\xi^h\|_{\phi_h}.\|\eta^h\|_{\phi_h}.\|\theta^h\|_{\phi_h}$

Now assume that $q, f \in Q^2(0,1)\ and\ P \in Q^3(0,1)\ where\ Q^s(0,1)$is the piece wise continuous differential function up to including $S\ also.$

Where $q = 2ce^{cx^2}, P = e^{cx^2}, f = f(x) = e^{-cx^2}\mu(1 + 4R\lambda_1 x^2)^{-\frac{3}{2}}$

The possible discontinuities being those of first kind at points $0 < Y_1 < Y_2 < \cdots < Y_m < 1$. We will assume everywhere in what follows that the set $\{Y_i\}_{i=1}^m$ belongs to the set of net points $\{x_k | k = 1,2,\dots,n-1\}$.

This assumption will be needed in analyzing the approximation error. From the assumption made it follows that the solution F of problem (17) will be continuous. While on each of the segment $[Y_i, Y_{i+1}])\_le\ on\ each\ of\ the\ segment\ \ the\ assllows\ that\ the\ set$ 5 $1,2,\dots,m-1$

The solution will have a fourth derivatives,

Let us now investigate the behavior of the components ξ^h, θ^h, η^h under the assumption that $h \ll 1$.

Where $\underset{0 \le k \le n-1}{Max}|x_{k+1} - x_k|$ $(*)$

Expanding the $(*)$ in to the Taylor series in the vicinity of the net points, it is not difficult to show that the components of these vectors are majorized in modulus by the corresponding components of the vector W^h.

Where $\{W^h\} = Nh. If\ x\ is\ one\ of\ the\ points\ Y_i\ (i = 1,2,\dots n) = M\left(\left|\Delta_{k+\frac{1}{2}} - \Delta_{k-\frac{1}{2}}\right| + h^2\right)\ otherwise$

$M\ and\ N\ are\ positive\ constants.$ Here we introduced the notation

$$\Delta_{k+\frac{1}{2}} = x_{k+1} - x_k$$

Let us assume that in the domain of definition of the solution there is a points of discontinuity of the coefficients.

$$x = x_i(1 \le i \le n)\ and\ \Delta x_{k+\frac{1}{2}} = \Delta x_{x-\frac{1}{2}}\ for\ k \ne 1$$

Form (27)

$$\|W^h\|_{\emptyset_h} = \left[\sum_{k=1,k\ne1}^{n-1}(w^h)_k^2\Delta x_k + (w^h)_1^2\Delta x_1\right]$$

Suppose that $h = \max\left\{\Delta x_k, 2\left(1 - x_{n-\frac{1}{2}}\right), 2x_{\frac{1}{2}}\right\}$

Taking the account of the relation $1 - h \le \sum_{k-1}^{n-1}\Delta x_{k-1}$ and using the above local estimating the terms W^h in the square norm. We obtain the estimate

$$\|W^h\|^2_{\emptyset_h} = \left[\sum_{k=1,k\neq 1}^{n-1}(w^h)^2_k \Delta x_k + (w^h)^2_1 \Delta x_1\right]$$

Now $\{W^h\}_k = N.h.$ If $x\, is$ one of the points $Y_i, i = 1,2,\dots,m.$

$$= M\left(\left|\Delta_{\left(k+\frac{1}{2}\right)} - \Delta_{\left(k-\frac{1}{2}\right)}\right| + h^2\right); otherwise\ M\ and\ N\ are\ positive\ constant\ or\ k \neq 1, x_{k-\frac{1}{2}} = x_{k+\frac{1}{2}}$$

$$\sum_{k=1}^{n-1}\Delta x_k < 1$$

$$\|W^h\|_{\emptyset_h} \leq (Mh^2)^2 + (Nh)^2 h$$

$$\|W^h\|_{\emptyset_h} \leq (M^2h^4) + (N^2h^2)h$$

$$\|W^h\|_{\emptyset_h} \leq M^2h^4 + N^2h^3$$

$$\|W^h\|_{\emptyset_h} \leq h^3(M^2h + N^2)\ where\ M\ and\ N\ are\ positive\ constant.$$

$$\|W^h\|_{\emptyset_h} \leq h^{\frac{3}{2}}\sqrt{M^2h + N^2}$$

$$\|W^h\|_{\emptyset_h} \leq h^{\frac{3}{2}}C, where\ C = \sqrt{M^2h + N^2}$$

Hence $\|W^h\|_{\emptyset_h} \leq Ch^{\frac{3}{2}}$

Where $C\ being\ positive\ constant.$ Hence we have the following estimate for the norms of approximation error of $\xi^h, \theta^h, \eta^h.$

$$\therefore (\|\xi^h\|, \|\eta^h\|, \|\theta^h\|) \leq Ch^{\frac{3}{2}} \qquad (31)$$

Where C being positive constant, independent of$h[Marchuk, 30]$. Provided one of the two conditions below are satisfied. Either the net is uniform on each of the intervals $[0, Y_1], [Y_1, Y_2], \dots, [Y_m, 1]$ or the net is quasi uniform.

$$i.e. The\ inequality\ \left|\Delta x_{k+\frac{1}{2}} - \Delta x_{k-\frac{1}{2}}\right| \leq Ch^2\ as\ h \to 0\ is\ violated\ only\ finitely\ many\ times. C > 0\ is\ constant.$$

Let us note that if the order of smoothness of any of the function $P, q\ and\ f\ is$ decreased by one the following estimate is obtained.

$$\max\{\|\xi^h\|. \|\eta^h\|. \|\theta^h\|\} < C_1.h$$

The difference scheme (28) which we have considered, is rarely used in practice the way it stand. Since the explicit integration of the function $p, q\ and\ f\ becomes$ very difficult.

As rule instead of (28) we used its simplified version.

$$(A^hF^h) = -\frac{1}{\Delta x_k}\left[P_{\left(k+\frac{1}{2}\right)}\frac{\left(F^h_{k+1} - F^h_k\right)}{\Delta x_{k+\frac{1}{2}}} - P_{\left(k-\frac{1}{2}\right)}\frac{\left(F^h_k - F^h_{k-1}\right)}{\Delta x_{k-\frac{1}{2}}} - (q\Delta x_k)_k F^h_k\right]$$

As earlier $P\ and\ q\ defined\ as$

$$P = e^{cx^2}\ and\ q = 2ce^{cx^2}$$

And $(f^h)_k = \frac{1}{\Delta x_k}[f\Delta x]_k = f_k$

$$= \frac{f_{k+\frac{1}{2}}\left(x_k - x_{k-\frac{1}{2}}\right) + f_{k-\frac{1}{2}}\left(x_{k+\frac{1}{2}} - x_k\right)}{x_{k+\frac{1}{2}} - x_{k-\frac{1}{2}}}\ where\ k = 1,2,3,\dots,(n-1) \qquad [Marchuk\ 14]$$

Where $f = e^{-cx^2}\mu(1 + 4R\lambda_1 x^2)^{-\left(\frac{3}{2}\right)}$

It turns out all the conclusion we have made with regard the size of the approximation on error still hold provided all the corresponding assumption on smoothness of parameter also remain unchanged. We will now turn to convergence properties $(28)\ and\ (29)$. Keeping the smoothness assumption on $P, q, and\ f\ we\ need$ only to prove the stability (28).

$$\|A\|^2 = \sup_{F \in F, F \neq 0}\frac{(AF, AF)}{(F, F)}$$

And then we used convergence theorem gives in $[Verma - 7]$ and it is state below for completeness.

1. Suppose that the difference scheme

$$A^h\emptyset^h = f^h\ in\ D_h$$
$$a^h\emptyset^h = g^h\ in\ \partial D_h$$

Approximate the initial problem,

$$A\emptyset = f\ in\ D_h$$
$$a\emptyset = g\ in\ \partial D_h\ to\ order\ n\ on\ the\ solution\ \emptyset$$

2. $A^h\ and\ a^h\ are\ linear\ operators.$
3. The difference scheme $A^h\emptyset^h = f^h\ in\ D_h$

$$a^h\emptyset^h = g^h\ in\ \partial D_h\ is\ stable.$$

i.e. ∃ positive constants $\bar{h}. c_1, c_2 \in h < \bar{h}, f_h, g_h \in G_h$

Then ∃ a unique solution $\emptyset_h$ of the problem 1 satisfying the inequality,

$$\|\emptyset^h\|_{\emptyset_h} \leq C_1\|f^h\|_{F_h} + C_2\|g^h\|_{G_h}$$

Then the solution of $\emptyset_h$ of the difference problem converges to the solution $\emptyset$ of initial problem.

i.e. $\lim_{h \to 0}\|(\emptyset)_h - \emptyset^h\|_{F_h} = 0$

And the following estimates of the rate of convergence is valid.

$$\|(\emptyset)_h - \emptyset^h\|_{F_h} \leq (M_1C_1 + M_2C_2)h^n$$

Where $M_1\ and\ M_2\ are\ constant.$

We first estimate the scalar product (F^h, f^h). By the Cauchy- Bunyakovsky inequality stated below.

$$\left|\sum_{n=1}^{\infty} S_n t_n\right| \leq \left(\sum_{n=1}^{\infty} S_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}}$$

Where $S_n\ and\ t_n$ are sequences. Reference by (Bokserman, Brownscombe,1 ,6). By using above statement, we have

$$(F^h, f^h) \leq \|F^h\|^{\frac{1}{2}}_{\emptyset_h}. \|f^h\|^{\frac{1}{2}}_{\emptyset_h} \qquad (32)$$

Where the scalar product is to be understood in following.

$$(\psi, \varphi) = \sum_{k=1}^{n-1}\Delta x_k \psi_k \varphi_k\quad where\ \psi, \varphi \in \emptyset_h$$

Let us investigate L.H.S. (32) in more detail. Since $q(x) \geq 0 and\ P(x) > 0\ by\ hypothesis\ we\ have,$

$$(F^h, f^h) = (F^h, A^hF^h)$$
$$= \sum_{k=1}^{n}\frac{F^h_k - F^h_{k-1}}{\int_{x_{k-1}}^{x_k} Pdx} + \sum_{k=1}^{n-1}\left(F^h_k\right)^2\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} q(x)dx$$
$$= \sum_{k=1}^{n}\frac{F^h_k - F^h_{k-1}}{\int_{x_{k-1}}^{x_k} e^{-cx^2}dx} + \sum_{k=1}^{n-1}\left(F^h_k\right)^2\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left(2ce^{cx^2}\right)dx$$
$$\geq p_0\sum_{k=1}^{n-1}\frac{\left(F^h_k - F^h_{k-1}\right)^2}{\Delta x_{k-\frac{1}{2}}}$$

$But\ p(x) = e^{cx^2}, q = 2ce^{cx^2}, p(0) = ce^0 = c$ [14]

$$(F^h, f^h) \geq p_0 \sum_{k=1}^{n-1} \frac{\left(F_k^h - F_{k-1}^h\right)^2}{\Delta x_{k-\frac{1}{2}}}$$

$$(F^h, f^h) > 0 \qquad (33)$$

This inequality follows from the fact that vector F^h is nonnull. Since it is the solution of the in homogeneous problem (28) with non-singular matrixA^h.

Noting that $F_0^h = 0$ *we may write,*

$$F_k^h = \sum_{j=1}^{k} \left(F_j^h - F_{j-1}^h\right)$$

$$= \sum_{j=1}^{k} \frac{\left(F_j^h - F_{j-1}^h\right)\sqrt{\Delta x_{j-\frac{1}{2}}}}{\sqrt{\Delta x_{j-\frac{1}{2}}}}$$

By the Cauchy- Bunyakovsky inequality for sum we obtain

$$\left(F_k^h\right)^2 = \left[\sum_{j=1}^{k} \frac{\left(F_j^h - F_{j-1}^h\right)\sqrt{\Delta x_{j-\frac{1}{2}}}}{\sqrt{\Delta x_{j-\frac{1}{2}}}}\right]^2$$

$$\leq \left[\sum_{j=1}^{k} \frac{\left(F_j^h - F_{j-1}^h\right)^2}{\left(\sqrt{\Delta x_{j-\frac{1}{2}}}\right)^2}\right]\left[\sum_{j=1}^{n-1} \left|\sqrt{\Delta x_{j-\frac{1}{2}}}\right|^2\right]$$

$$\leq \left[\sum_{j=1}^{k} \frac{\left(F_j^h - F_{j-1}^h\right)^2}{\Delta x_{j-\frac{1}{2}}}\left[\sum_{j=1}^{n-1} \Delta x_{j-\frac{1}{2}}\right]\right]$$

$$\left(F_k^h\right)^2 \leq \left[\sum_{j=1}^{k} \frac{\left(F_j^h - F_{j-1}^h\right)^2}{\Delta x_{j-\frac{1}{2}}}\right] \left(\because \sum_{j=1}^{n-1} \Delta x_{j-\frac{1}{2}} \leq 1\right) \qquad (33)$$

From (32) *and* (33) we have

$$(F^h, f^h) \leq \|F^h\|_{\emptyset_h} \|f^h\|_{\emptyset_h}$$

$$\|F^h\|_{\emptyset_h} \|f^h\|_{\emptyset_h} \geq (F^h, f^h)$$

$$\geq \left[\sum_{j=1}^{k} \frac{\left(F_j^h - F_{j-1}^h\right)^2}{\Delta x_{j-\frac{1}{2}}}\right]$$

$$\geq \sum_{j=1}^{n-1} \left(F_k^h\right)^2 \Delta x_k$$

$$\|F^h\|_{\emptyset_h} \|f^h\|_{\emptyset_h} \geq \|F^h\|_{\emptyset_h}^2$$

$$\|f^h\|_{\emptyset_h} \geq \|F^h\|_{\emptyset_h}$$

$$\|F^h\|_{\emptyset_h} \leq \|f^h\|_{\emptyset_h}$$

This inequality prove the stability of the difference algorithm (by 32 *to* 34).

Using the convergence theorem with the norm (27) we obtained the estimate as below.

Here $\|F^h\|_{\emptyset_h}^2 = \sum_{k-1}^{n-1} \left(F_k^h\right)^2 \Delta x_k$

Using the convergence theorem which was earlier used on . Again we use the convergence theorem with assume values C_1, M_1 *and* h *as constant then,*

$$\|F^h\|_{\emptyset_h}^2 \leq C_1 M_1 h^3 \left(\because \sum_{k=1}^{n-1} \Delta x_k < h \text{ and } \sum_{k=1}^{n-1} \left(F_k^h\right)^2 < C_1 M_1 h^2\right)$$

$$\|F^h\|_{\emptyset_h} \leq \sqrt{C_1 M_1 h^3}$$

$$\leq \sqrt{C_1 M_1} h^{\frac{3}{2}}$$

$$\|F^h\|_{\emptyset_h} \leq k h^{\frac{3}{2}} \text{ where } k = \sqrt{C_1 M_1} \text{ is constant.}$$

$$\varepsilon^h = (F)_h - F^h$$

$$\|\varepsilon^h\|\emptyset_h \leq k h^{\frac{3}{2}} \quad \varepsilon^h = (F)_h - F^h$$

where $k \geq 3C$ *is positive constant.* By drawing certain networks analog of the Imbedding theorem we can clarify the estimate

$$\|\varepsilon^h\|\emptyset_h \leq k h^{\frac{3}{2}}$$

First we note that $F_0^h = F_n^h = 0$

$$\left(F_k^h\right)^2 \leq \left[\sum_{j=1}^{n} \frac{\left(F_j^h - F_{j-1}^h\right)^2}{\Delta x_{j-\frac{1}{2}}}\right]$$

$$\leq \left[\sum_{j=1}^{n} \frac{\left(F_j^h - F_{j-1}^h\right)^2}{\left(\Delta x_{j-\frac{1}{2}}\right)^2} \Delta x_{j-\frac{1}{2}}\right]6$$

$$\leq \sum_{j=1}^{n} \left[\left[\frac{\left(F_j^h - F_{j-1}^h\right)}{\Delta x_{j-\frac{1}{2}}}\right]^2 \Delta x_{j-\frac{1}{2}}\right]$$

If $C_1 \leq \left[\frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j}\right] \leq C_2; C_1, C_2 > 0$ are constant and independent of j. We have

$$\left(F_k^h\right)^2 \leq \sum_{j=1}^{n} \left[\frac{\left(F_j^h - F_{j-1}^h\right)}{\Delta x_{j-\frac{1}{2}}}\right]^2 \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} \Delta x_j$$

$$\leq C_2 \sum_{j=1}^{n} \left[\frac{\left(F_j^h - F_{j-1}^h\right)}{\Delta x_{j-\frac{1}{2}}}\right]^2 \Delta x_j$$

$$\left(F_k^h\right)^2 \leq C_2 (\|F^h\|^2)_{w_{1},h}^0$$

From this we obtain the following relation for the net function (the net analog of imbedding $W_2^1(0,1)$ *in* $C(0,1)$ in one dimension case)

Where $W_2^1(0,1)$ is a Sobolovespace $(0,1)$. Which is Sobolovespace function in $W_2^1(D)$ *that vanish on* ∂D

$$\|F^h\|_{\emptyset_h} = \underset{1 \leq j \leq n-1}{Max} \left|F_k^h\right|$$

$$\|F^h\|_{\emptyset_h} \leq C\, \|F_h\|_{W_{1,h}}^0 \text{ where } C = constant < \infty$$

We also apply the later inequality to obtain a more precise estimate of the error.

$$\varepsilon^h = (F)_h - F^h$$

Then we write an identity

$$A^h \varepsilon^h = \xi^h + \eta^h + \theta^h$$

Then we take the scalar product with ε^h

$$(A^h \varepsilon^h, \varepsilon^h) = (\xi^h + \eta^h + \theta^h, \varepsilon^h)$$

From (33)

$$(A^h \varepsilon^h, \varepsilon^h) \geq \sum_{k=1}^{n} \left[\frac{\left(F_k^h - F_{k-1}^h\right)}{\Delta x_{k-\frac{1}{2}}}\right]^2$$

$$(A^h \varepsilon^h, \varepsilon^h) \geq C_1 \|\mathcal{E}^h\| W_{1,h}^0$$

$$i.e\ |(\xi^h + \eta^h + \theta^h), \varepsilon^h| = \left|\sum_{k=1}^{n-1} \Delta x_k\ (\xi^h + \eta^h + \theta^h)\varepsilon^h\right|$$

$$\le \|\varepsilon^h\|_{\emptyset_h} \sum_{k=1}^{n-1} \Delta x_k\ \|\xi^h + \eta^h + \theta^h\|$$

$$|(\xi^h + \eta^h + \theta^h), \varepsilon^h| \le \|\varepsilon^h\|_{\emptyset_h} \|\xi^h + \eta^h + \theta^h\|_{L_1,h}$$

We have

$$\|\varepsilon^h\|_{W^0_{1,h}} \le C\|\xi^h + \eta^h + \theta^h\|_{L_1,h}$$

Drawing on the above imbedding theorem, we obtain the inequality

$$\|\varepsilon^h\|_{W^0_{1,h}} \le C\|\xi^h + \eta^h + \theta^h\|_{L_1,h}$$

But we have that necessary smoothness of the solution and the initial data the quasi-uniform of the net

$$\|\xi^h + \eta^h + \theta^h\|_{L_1,h} < 3Nmh^2 + CmH^2 \sum_{k=1}^{n-1} \Delta x_k \quad where\ C = constant < \infty$$

For the sufficient small $h\ and\ m < \infty$ we get the desire estimate

$$\|\varepsilon^h\|_{C_h} \le C\|\varepsilon^h\|^0_{W_{2,h}} < 0(h^2) \qquad (34)$$

III. CONCLUSION

We looked at the flow of two immiscible liquids in a broken porous medium in this problem. The flow in fractured medium has been changed from a nonlinear differential system to an ordinary differential equation, which has then been translated to a diffusion equation. Confluent hyper-geometric series and an integral form are used to get the solution. The equation (34) gives the solution of our diffusion equation and we established the stability and found the estimates of them.

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