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Fractional Order Calculus and Derivative Implementation

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ABSTRACT- The concept of fractional calculus dates back to the early days of calculus and may be traced back to Arbogast's publications in the 1800s. Multiple derivatives are defined in this situation by powers of D that are logically discovered to be integral in nature. However, this gave birth to the idea of evaluating this operator's fractional power and attempting to determine its equivalent form or operating it on some function in a meaningful way.This part of calculus is classified as special calculus, and it did not find much application in engineering until the advent of electronic computers and control systems, where it began to show its capability, such as increasing the number of control parameters in a PID control system, which essentially increases its ability to be optimised, albeit at a higher complexity. This project begins with introducing the fundamentals of fractional calculus, followed by fractional derivatives of standard functions and their interpretation, and then provides fractional calculus applications in the context of electronics.

KEYWORDS- Fractional order, PID, Control Systems.

I. INTRODUCTION

Multiple derivatives are defined in this situation by powers of D that are logically discovered to be integral in nature. The notion of examining this operator's fractional power and attempting to determine its equivalent form or to operate it on some meaningful function. This part of calculus is classified as special calculus, and it did not find much application in engineering until the advent of electronic computers and control systems, where this concept began to demonstrate its capability, such as increasing the number of control parameters in a PID control system, which essentially increases its ability to be optimised, albeit at a higher complexity.

The fractional calculus is understandable in the sense that the derivative is not computed in the sense that it is a tangent at a point, as in the regular calculus. The meaning is still hazy here, however it may be thought of as having numerous points evaluated in the operation rather than just one. This offers the fractional calculus the benefit of simultaneously functioning on dispersed data, but it has yet to be understood and a meaning created from it.

A. Methodology

The project begins by introducing the notion of fractional calculus, as well as operations on standard functions. This Following that, a tiny discriminant is built in the analogue domain, which is used in various systems such as filters. Following that, PSE and CFE generating methods are compared. Following that, a quick overview of fractional differentiation in Z transform is provided. Finally, a fractional order filter and a multiphase oscillator constructed with fractional calculus are described.

The programmed utilized for PSE and CFE is 'Maple,' and MATLAB is used to acquire response of resulting functions. To build a multiphase oscillator, first calculate the parameters and then simulate them in NI Multisim.

II. LITERATURE REVIEW

Fractional Calculus is a Mathematical Analysis tool used to examine integrals and derivatives of any order, both fractional and real. Many scientists are unfamiliar with fractional integrals and derivatives, and they have only recently been employed in a pure mathematical context. However, integrals and derivatives have been used in a variety of scientific applications throughout the previous few decades.

Leibniz's discussion with L'Hospital in 1695 [1] introduced the concept of pro rank (pro rata) variations. The theory of fractals was designed primarily as a philosophical subject of mathematics for many periods after then. Furthermore, this powerful mathematical tool has recently found applications in numerous of fields, including linear programming [6], supercapacitors [7], brain modeling [8], and more — see [9] for a fundamental examination as well as more implications of nonlinear problems. As representations become more widely used, it is sense to investigate the development of effective and efficient countable filters in [10], the study implies a Taylor provided the impetus of the likely to be true to be differentiated while constructing discrete partial differential equation filters. The developers of [11] use a similar tactic, choosing a Newton general form for the Finite volume method. See also [12] for a larger parametric study.

III. METHODOLOGY

A. Fractional Calculus

As previously covered in prior chapters, the beginnings of fractional calculus now allow us to go on to attempting to derive meaning from it by solving and comprehending it.

B. Origins and Basics

Fractional calculus is derived from the 'D' notation for the differential operator, Da, where a denotes the order of derivative. This term is often used to define a series of derivatives.

$$D^n(f) = (\underbrace{D \circ D \circ D \circ \cdots}_{n})(f) = \underbrace{D(D(D\cdots}_{n}(f)$$

Where, *D* is equivalent to

$$Df(x) = \frac{d}{dx} f(x) \ ,$$

However, this does not limit us to integer power; we may also test a fractional power of D, such as,

$$\sqrt{D} = D^{\frac{1}{2}}$$

This notation, on the other hand, is not obvious.When attempting to describe a comparable operation on integrals, we may utilise the Cauchy integral to aid in the solution of a fractional integral; but, due to the inclusion of the gamma function in the formulation, we cannot use it for negative values or derivatives. In that sense, it becomes extremely tough to solve.

All true numbers of n are included in the specification of dny/dxn. Considering the function f (x), which is established for x larger than zero. Make a second derivative with values ranging from 0 to x. It's referred to as,

$$(Jf)(x) = \int_0^x f(t)\, dt \ .$$

Repetition of this procedure yields

$$\left(J^2 f\right)(x) = \int_0^x (Jf)(t)\, dt = \int_0^x \left(\int_0^t f(s)\, ds\right) dt,$$

This can be continued indefinitely.

For periodic integrate, Cauchy's theorem, therefore

$$\left(J^n f\right)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\, dt,$$

results in a straightforward extension for real n

The ionization tool is useful for quintic operating company tasks because it eliminates the basic function's equations to describe.

$$\left(J^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \ .$$

This is an action with a clear definition.

The semigroup feature of fractional differintegral operators of this concept is also useful.

$$\left(J^\alpha\right)\left(J^\beta f\right)(x) = \left(J^\beta\right)\left(J^\alpha f\right)(x) = \left(J^{\alpha+\beta} f\right)(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^x (x-t)^{\alpha+\beta-1} f(t)\, dt \ .$$

$$\begin{aligned}\left(J^\alpha\right)\left(J^\beta f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left(J^\beta f\right)(t)\, dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^t (x-t)^{\alpha-1} (t-s)^{\beta-1} f(s)\, ds\, dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s) \left(\int_s^x (x-t)^{\alpha-1} (t-s)^{\beta-1}\, dt\right) ds\end{aligned}$$

As a result, we altered the validity and the reliability in the last phase and retrieved the f (s) ingredient from the t assessment. t = s + (x s) r

$$\left(J^\alpha\right)\left(J^\beta f\right)(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x (x-s)^{\alpha+\beta-1} f(s) \left(\int_0^1 (1-r)^{\alpha-1} r^{\beta-1}\, dr\right) ds$$

That beta form, which has the set of criteria, is the innermost exponential.:

$$\int_0^1 (1-r)^{\alpha-1} r^{\beta-1}\, dr = B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Replacing the original equations with a new one

$$\left(J^\alpha\right)\left(J^\beta f\right)(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^x (x-s)^{\alpha+\beta-1} f(s)\, ds = \left(J^{\alpha+\beta} f\right)(x)$$

Changing constantly and achieves the evidence by demonstrating that the order in which the J operator is employed is unimportant.

In addition, the Riemann–Liouville fractional integral is defined. "The Riemann–Liouville integral, which is basically what has been explained above, provides the traditional form of fractional calculus." The Weyl integral is the theory for periodic functions (which includes the "border condition" of recurring after a period). It is based on the Fourier series and necessitates the disappearance of the constant Fourier coefficient. Upper and lower Riemann-Liouville integrals exist. The integrals are defined for the interval [a, b] as"

$${}_aD_t^{-\alpha} f(t) = {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

$${}_tD_b^{-\alpha} f(t) = {}_tI_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau$$

Where the former is true when t > a and the later is true when t b.

There have been various interpretations of fractional integrals over time, but we will not go into them here

IV. SYSTEM ARCHITETURE

The phase shift behaviour of fractional derivatives of sinusoids is shown in figure 1 below for sin(x), where $\alpha = 0.25, 0.5, 0.75$

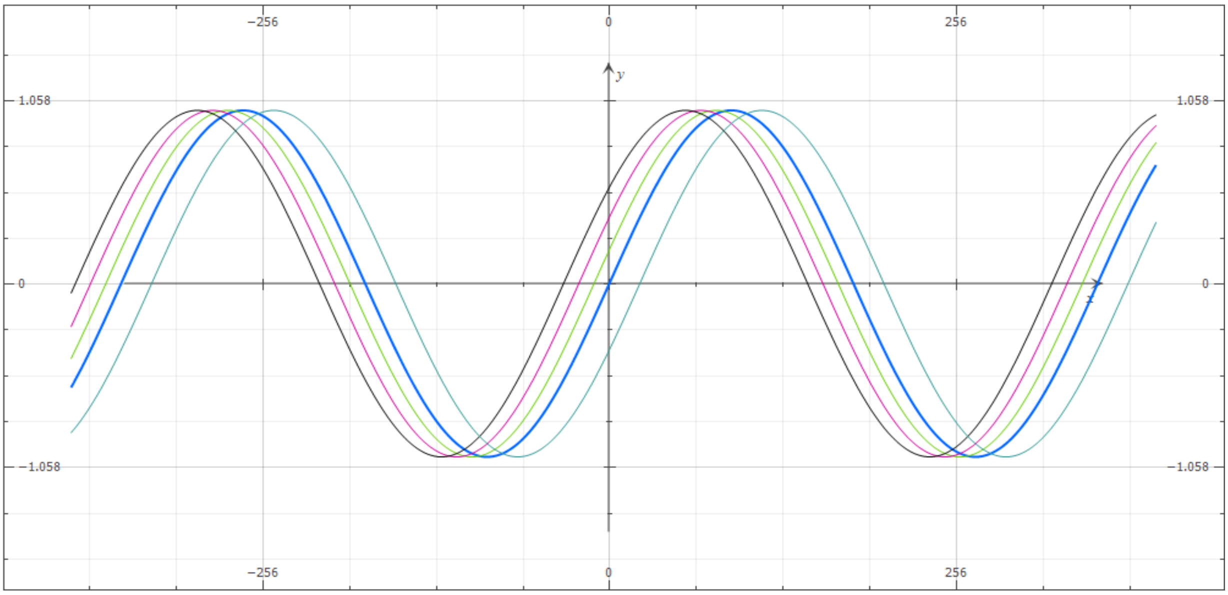

Figure 1: The phase shift behaviour of fractional derivatives

The legend is as follows, and each phase shift is clearly a fractional derivative of sin(x) in blue.

---- $y = sin(x)$, ---- $y = sin\left(\left(\frac{180}{\pi}\right)\right)$,

---- $y = sin\left(\left(\frac{180}{\pi}\right)\right)$, ---- $y = sin\left(\left(\frac{180}{\pi}\right)\right)$,

---- $y = sin\left(\left(\frac{180}{\pi}\right)\right)$

Because the angles are in radians, the 180-conversion term is used for graphing.

A. Analog interpretation of Fractional Derivative

In mathematics, analogue domain or frequency domain is frequently used to try an alternate technique to solving systems, which is often easier. It is also used in engineering to determine system performance and to develop systems. There are various techniques to go to frequency domain, but only two are commonly utilised in engineering. Fourier and Laplace transform. For the sake of attempting to make sense of fractional derivatives, the latter makes sense because it is frequently employed to aid in the solution of differential equations by transforming them into algebraic problems that are much easier to solve.

We begin by defining the derivative and integral in the Laplace domain. We know that the derivative is changed to's' and the integral is translated to '1/s'. By increasing the power of s, this notation may be extended to numerous derivatives and integrals.

$$\frac{1}{s^a}$$

Where, $a \in Z$. If $a > 0$ then it is an integrator, if $a < 0$ then differentiator.

However, in this situation, it should be emphasised that it only works if the original requirements are not met.

We may expand the domain of a to rational numbers, as we can with 'D' notation. This results in the realisation of fractional derivative and integral in analogue domain.Here now a is replaced with α,

$$\frac{1}{s^\alpha}$$

Where, $\alpha \in R$

It would be useful to be able to acquire the step response, bode plot, of this extended operator in order to observe its behaviour. However, the program we use for this is MATLAB, which does not allow fractional derivatives by default and hence cannot get these results.

To get around this issue, we apply conventional expansions to get integral order equivalent transfer functions. This is accomplished in the following manner:

Evaluation of fractional derivative using PSE and CFE

To evaluate the $s^{\pm\{\alpha\}}$ in Maple using CFE and PSE, we use the following form,

$$(1 + x)^\alpha$$

Where, $x = s - 1$

The original equation is then evaluated using expansion procedures, and the x is replaced with the supplied value again, and the problem is simplified. This results in an integral order equivalent transfer function, which can be calculated for different alpha values and visualised in MATLAB using the control systems module.

B. Comparison of CFE and PSE

After reduction, it is obvious that CFE gives a superior approximation than PSE in this example since CFE has less complexity and less variance than PSE. The MATLAB analysis is provided later down.

C. MATLAB

The following data shows the step response of the analogue version of fractional derivative at various alpha values.

The fractional derivative response obtains step and bode responses using the MATLAB control systems tool box. This is done with the following values: = 0.7, 0.75, 0.8, 0.85, 0.9, 0.95. Using the CFE coefficients to define a transfer function;

```
x = .7;          % replacing α with x for MATLAB
P0 = x^2 + 3*x + 2;
P1 = 8 - 2 * x^2;
P2 = x^2 - 3*x + 2;
A= [P2 P1 P0];
B= [P0 P1 P2];
C=tf (A, B)
```

The resultant transfer function becomes
C =
0.39 s^2 + 7.02 s + 4.59

4.59 s^2 + 7.02 s + 0.39
The step response is shown for the system in figure 2

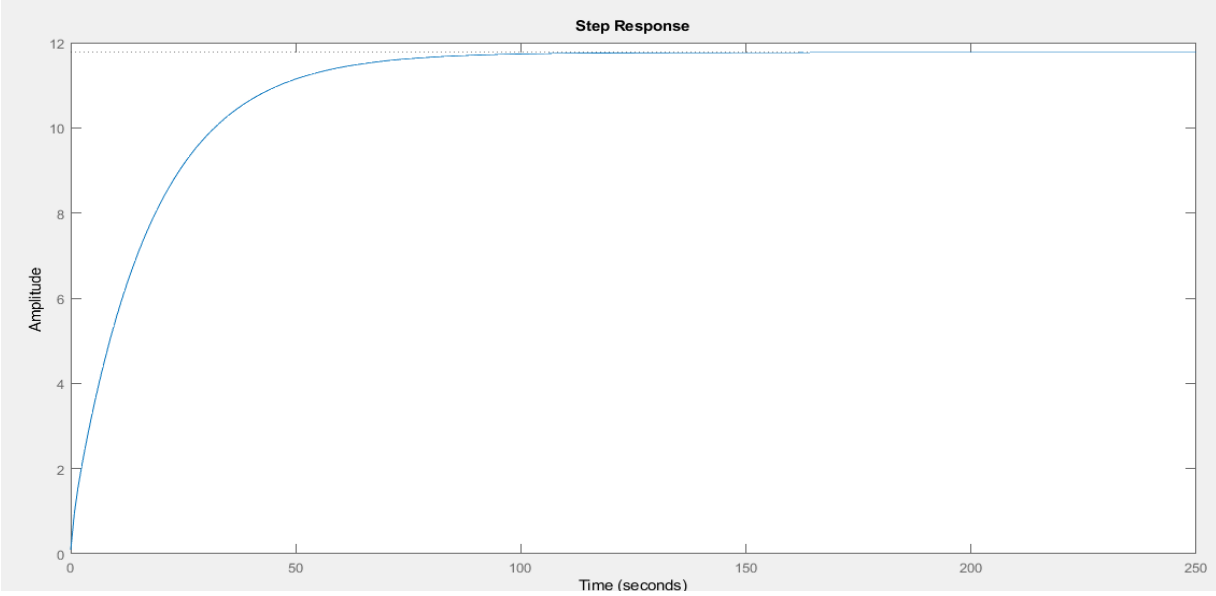

Figure 2: Step response for the System

Also, the bode plot in figure 3 for the same is;

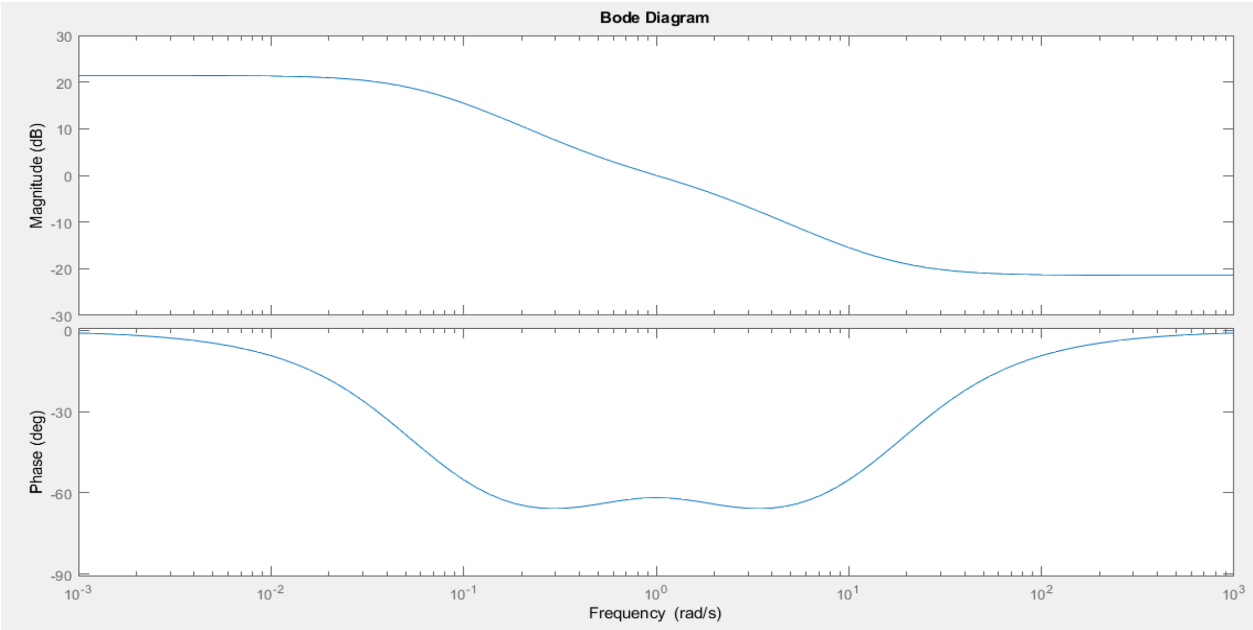

Figure 3: Bode Diagram

For α = 0.75, C =
0.3125 s^2 + 6.875 s + 4.813

4.813 s^2 + 6.875 s + 0.3125
STEP AND BODE in figure 4 and 5

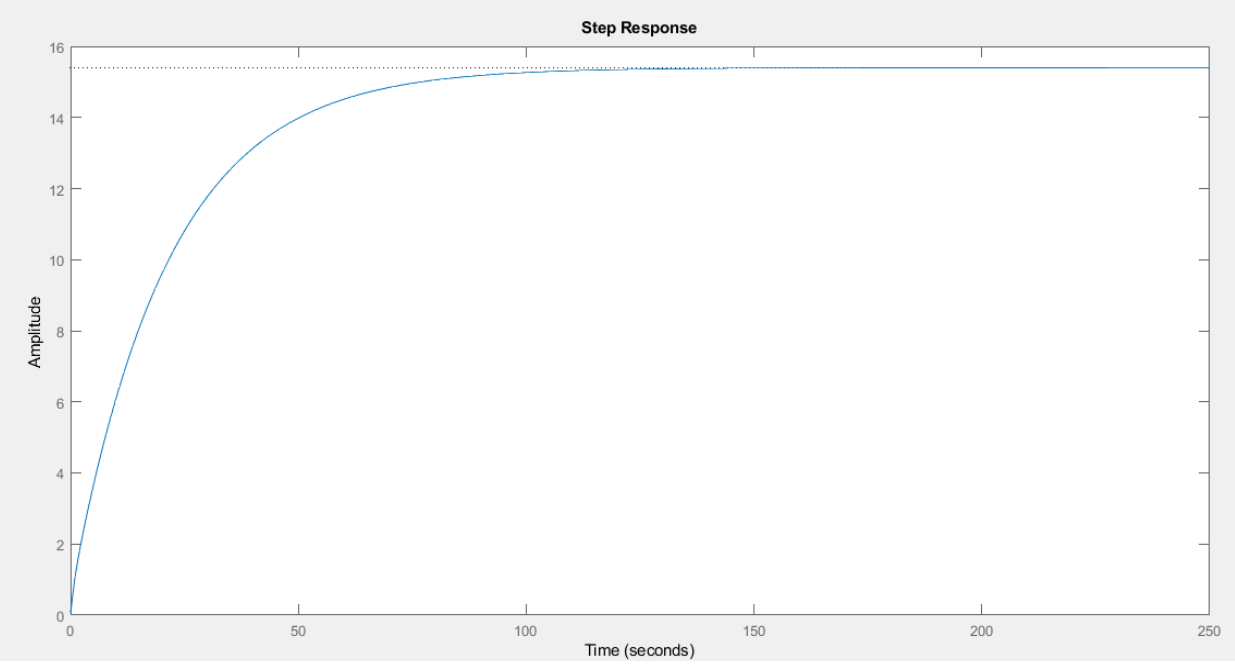

Figure 4: Step response α =0.75

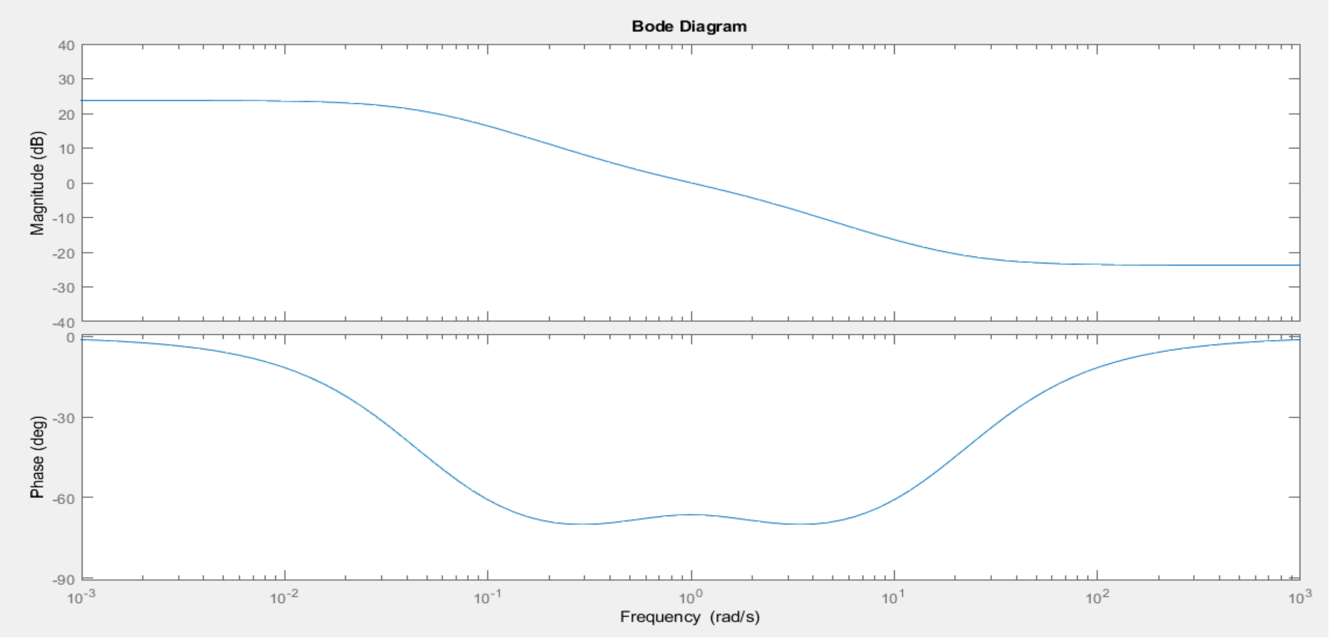

Figure 5: For α=0.75 bode response

For α = 0.8, C =

0.24 s^2 + 6.72 s + 5.04

5.04 s^2 + 6.72 s + 0.24

STEP AND BBODEin figure 6 and 7

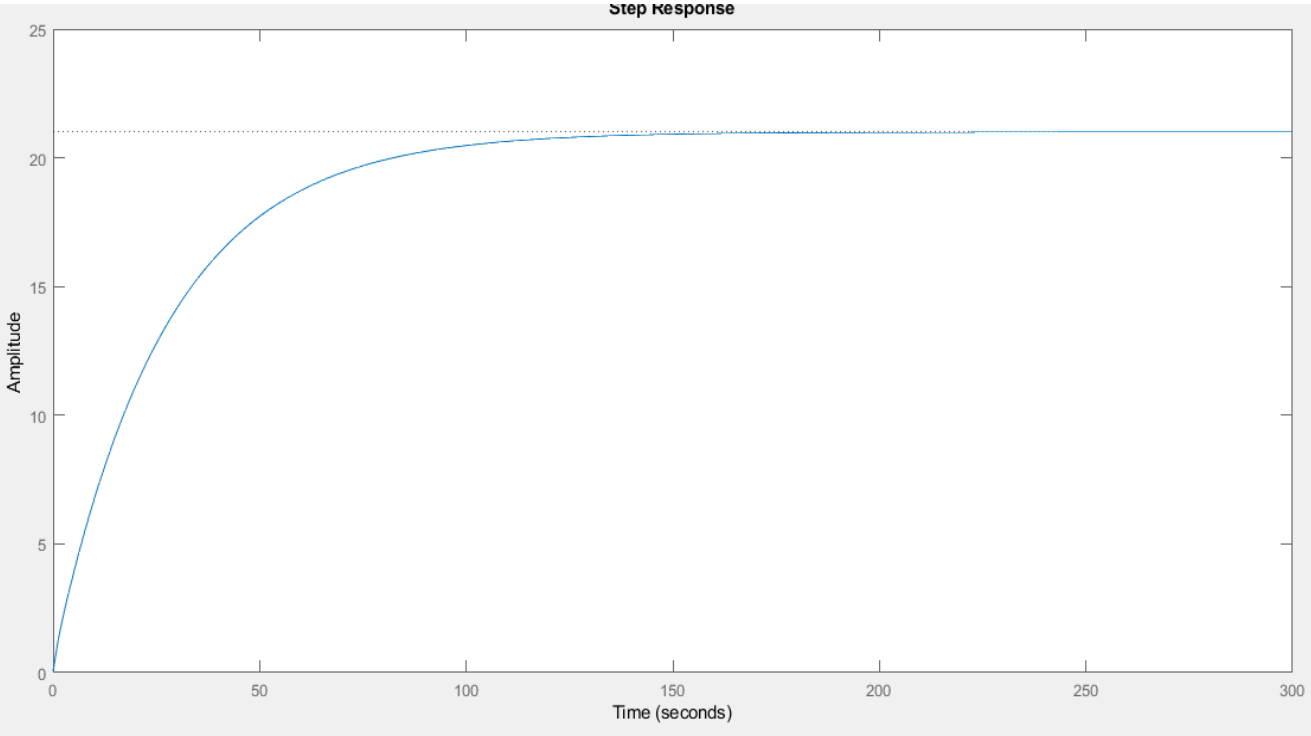

Figure 6: step for α = 0.8

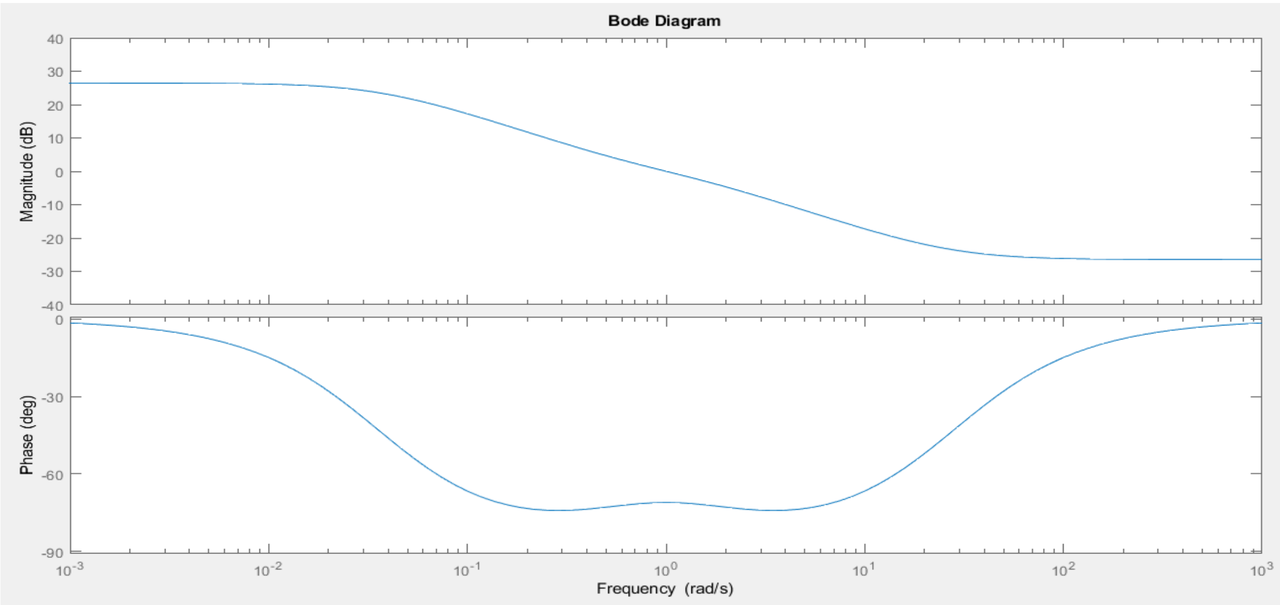

Figure 7: Bode for α = 0.8

V. RESULTS

A. Moving to Discreet domain (Z-domain)

Moving to the discrete domain with the Z-transform is possible by sampling the Laplace transform. This is advantageous in signal processing. For fractional derivatives, there are two methods: direct and generating function, as demonstrated below.

The generating function technique begins with the specification of the Z-transform and proceeds to find its derivative, resulting in a fractional power of z. This is the derivative property of the Z-transform, as seen below.

The generating function for Z-transform is given as;

$$Zx[n] = \sum_{k=-\infty}^{\infty} x[n]z^{-k}$$

To obtain a derivative using the same, we differentiate the both sides to obtain,

$$\frac{d}{dz}X(z) = \sum_{k=-\infty\to\infty} x[n]\frac{d}{dz}z^{-k}$$

$$\sum_{k} (-k)x[n]z^{-k-1} = -\frac{1}{z}\sum_{k} nx[n]z^{-k}$$

Successive differentiations are therefore denoted by the standard Z transform of the provided function multiplied by na/za, where an is the order of differentiation. Similarly to the continuous derivative, the domain of order an is extended to that of real numbers in this case, and we obtain a fractional derivative in the discrete domain.

$$D^{\alpha}\{Z(x[n])\} = \sum_{k=-\infty}^{\infty} x[n]\{D^{\alpha}z^{-k}\}$$
$$= (-1)^{1+\lceil\alpha\rceil}\left(\frac{k}{z}\right)^{\alpha} Z(x[n])$$

When utilising computers, the alternate approach for generating the z transform is more feasible. This approach simply samples the Laplace domain transfer function and transforms it to the z domain using MATLAB's c2d function, as illustrated below. The discrete answer is also presented.

This is a 4th order example system built in MATLAB.

```
x = -.5; % 0.5 order integral
P0 = x^4 + 10*x^3 + 35*x^2 + 50*x + 24;
P1 = -4*x^4 - 10*x^3 + 40*x^2 + 320*x + 384;
P2 = 6*x^4 - 155*x^2 +864;
P3 = -4*x^4 + 20*x^3 + 40*x^2 - 320*x + 384;
P4 = x^4 - 10*x^3 + 35*x^2 - 50*x + 24;
A= [ P0 P1 P2 P3 P4];
B= [ P4 P3 P2 P1 P0];
C=tf(A,B)
step(B)
bode(B)
B= c2d(C,0.001,'tustin')
step(B)
bode(B)
```

B. OUTPUT

```
C =
  6.563 s^4 + 235 s^3 + 825.6 s^2 + 551.3 s + 59.06
  59.06 s^4 + 551.3 s^3 + 825.6 s^2 + 235 s + 6.563
```

Continuous-time transfer function shown in figure 8.

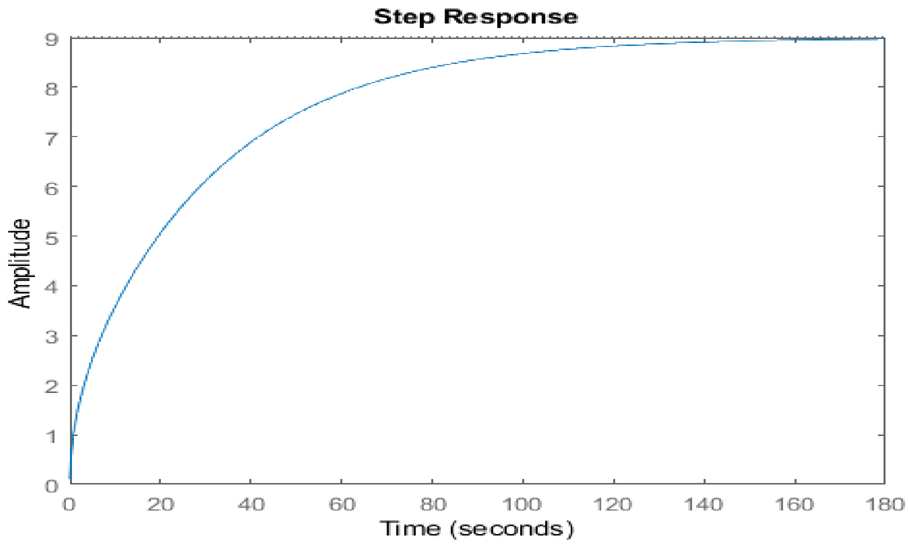

Figure 8 Transfer Function

For C

Bode Diagram for phase and magnitude can be seen in figure 9.

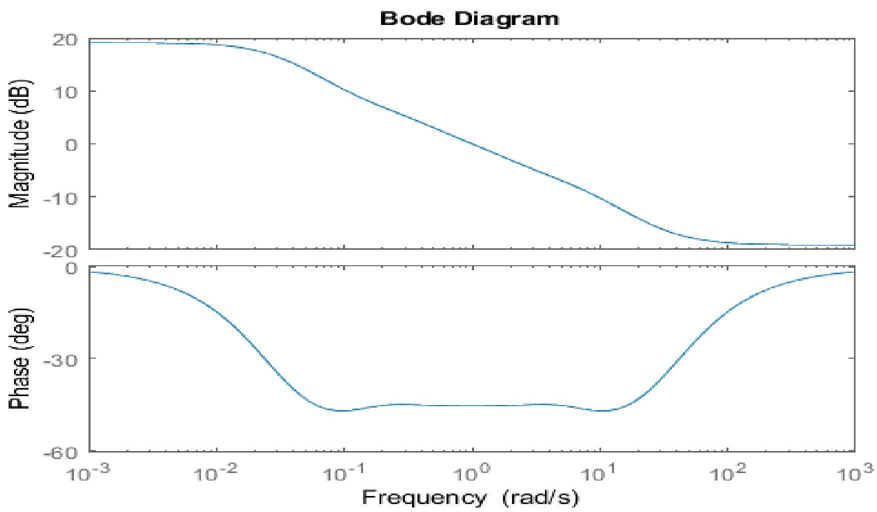

Figure 9: Bode Diagram for phase and magnitude

For C
B =
0.1126 z^4 - 0.4463 z^3 + 0.6636 z^2 - 0.4384 z + 0.1086
z^4 - 3.991 z^3 + 5.972 z^2 - 3.972 z + 0.9907

Sample time: 0.001 seconds
Discrete-time transfer function.
As can be seen in the figure 10 system B, it has now been turned to discreet equivalent.

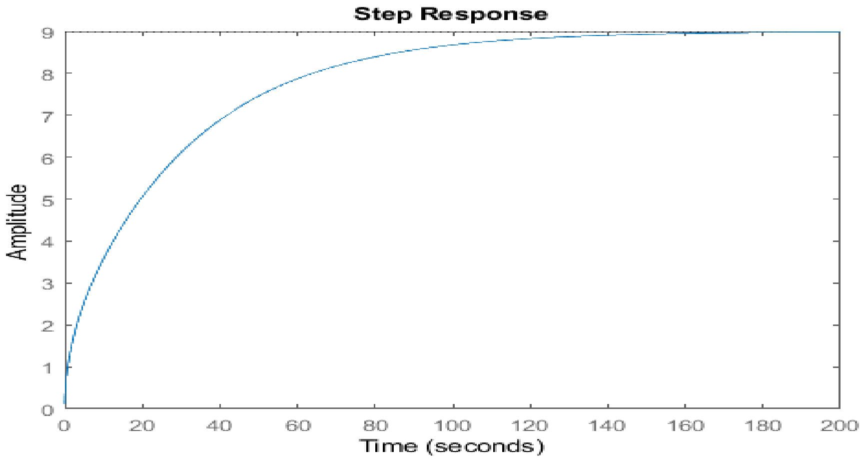

Figure 10 Step response for B

For B, (it is similar to C as seen in figure 11 is expected, because it is just a discreet version.)

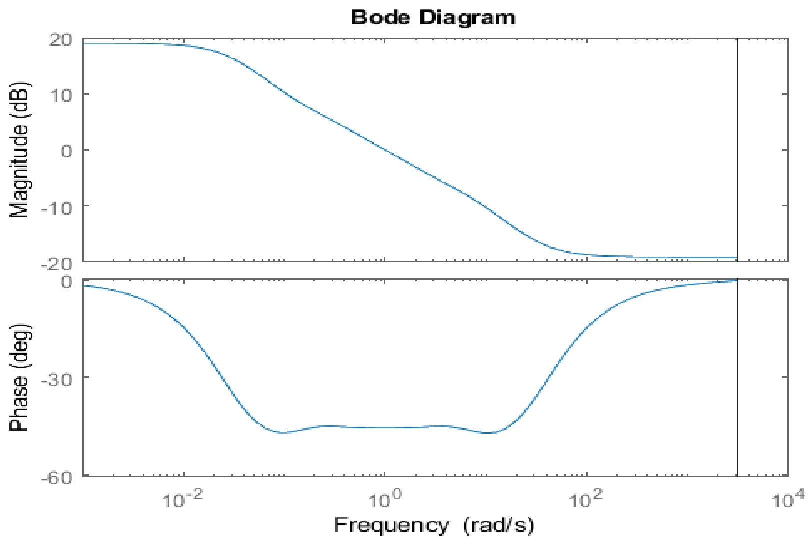

Figure 11: Step response for B

For B, (this is also similar to its continuous equivalent as it is generated from the same.)

C. Fractional Derivative and Integral

As we have shown in figure 8, we may use the's' notation to denote the fractional derivative and integral in the analogue domain, and we have also seen how it behaves in comparison to the integer order differential operators. In addition, we created a way for using the fractional differential operator in the z-domain. These approaches provide us more control over systems that are normally represented in the analogue realm by the derivative and integral notation of s, notably energy storage components such as capacitors and inductors. This is a highly strong tool, and its investigation is and will be beneficial in solving engineering challenges, including those in electronics, whether through simulation or actual system design. Observation for the values of S can be seen in figure 12.

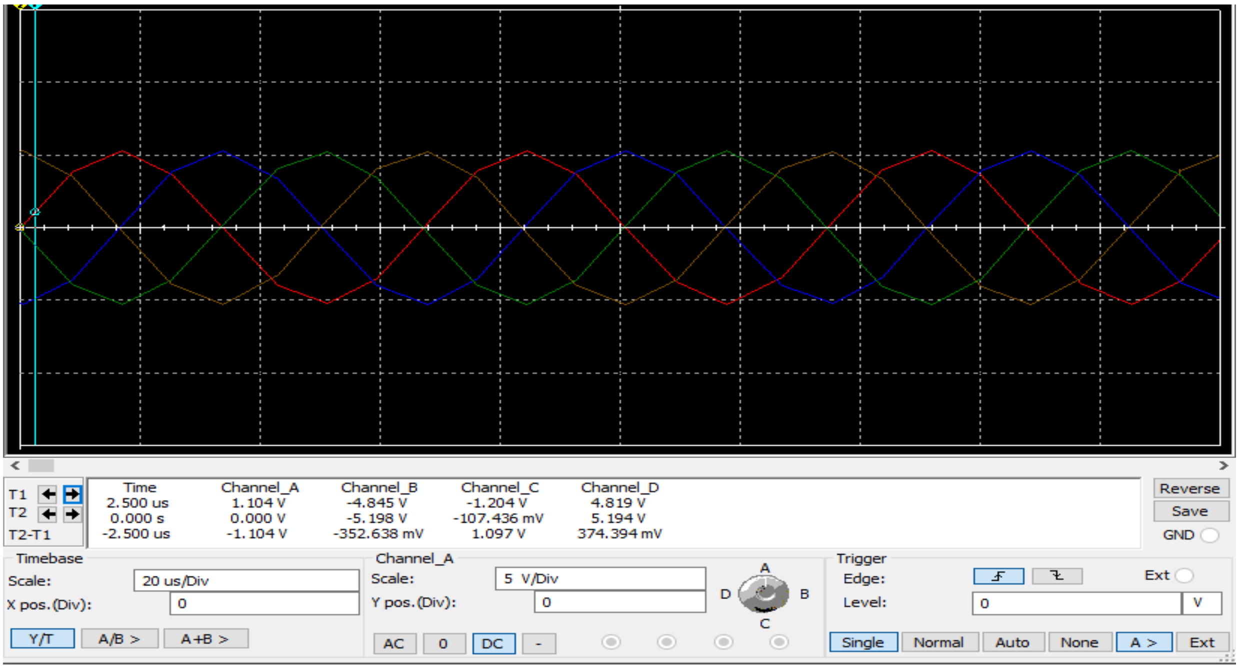

Figure 12: Observation for the values of S

s can be observed, the waveforms are 90 degrees out of phase with one another.

The second fractional alpha situation is now presented and simulated.

Phase shifts of 1=2=3=4 =0.8 or 72o 100 krad/sec = o Also, R1 = R2 = R3 = R4 = R, C1 = C2 = C3 = C4 = C, and the feedback constants are a0 = 1.618034; a1 = -1; a2 = 1.

As a result, the characteristic equation in this example is;

$$s^{3.2} + \frac{a_3}{RC}s^{2.4} + \frac{a_2}{R^2C^2}s^{1.6} + \frac{a_1}{R^3C^3}s^{0.8} + \frac{a_o}{R^4C^4} = 0$$

For this case the values of R_n and C_n are R1 = 193 kohm; R2 = R3 =10 kohm, R4 = 11.6 kohm and C1 = C2 = C3 = C4 = 10 nF.

After putting in the values, the following results are obtained shown in figure 13.

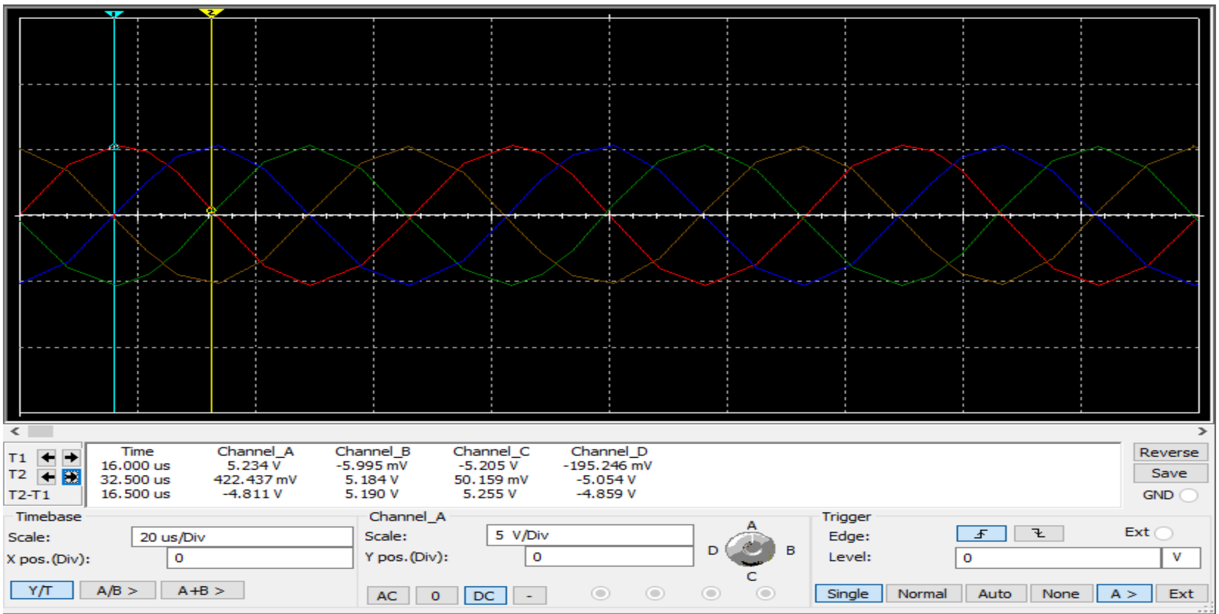

Figure 13: Results obtained

VI. CONCLUSION

The applications section clearly indicates that the system provides for more control and the design of systems that will pave the way for more sophisticated technologies or improve existing ones.

To summarise, the genuine advantage of fractional calculus will only grow in the future as it allows us to attain finer control and greater functionality by using electronics and computer areas ranging from control systems to digital signal processing or oscillator designs.

There is a lot of work to be done, such as trying to understand and develop applications for fractional calculus in many other fields of technology, such as simulations and the design of real-world hardware systems.

REFERENCES

[1] G. Gorenflo, Fractional calculus: some numerical methods. New York: Springer-Verlag, 1997, pp. 227–290

[2] G. Leibniz, Mathematische Schiften. Georg Olms Verlagsbuchhandlung, Hildesheim, 1962.

[3] K. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: John Wiley & Sons Inc., 1993.

[4] K. Oldham and J. Spanier, The Fractional Calculus. New York – London: Academic Press, 1974.

[5] K. Oldham, "A signal-independent electroanalytical method," Anal. Chem., vol. 44, no. 1, pp. 196–198, 1972.

[6] M. Ortigueira, "Introduction to fractional linear systems. Part 1: Continuous-time case," IEE Proceedings Vision, Image, Signal Processing, vol. 147, no. 1, pp. 62–70, 2000.

[7] M. Ortigueira, "Introduction to fractional linear systems. Part 2: Discrete-time case," IEE Proceedings Vision, Image, Signal Processing, vol. 147, no. 1, pp. 71–78, 2000.

[8] Oustaloup, "From fractality to non-integer derivation through recursivity, a property common to these two concepts: A fundamental idea for a new process control strategy," in Proceedings of the 12th IMACS World Congress, 1988, pp. 203–208.

[9] Podlubny, Fractional Differential Equations. San Diego: Academic Press, 1999.

[10] S. Samadi, M. O. Ahmad, and M. Swamy, "Exact fractional-order differentiators for polynomial signals," IEEE Signal Processing Letters, vol. 11, no. 6, pp. 529–532, 2004.

[11] T. Anastasio, "The fractional-order dynamics of brainstem vestibulo-oculomotor neurons," Biological Cybernetics, vol. 72, pp. 69–79, 1994.

[12] Tseng, "Design of fractional order digital FIR differentiators," IEEE Signal Processing Letters, vol. 8, no. 3, pp. 77–79, 2001.