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Research Article

An Efficient Analytical Solution for Fractional Order Cancer Model; Laplace Transformation

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ABSTRACT

This research aims to determine the approximate analytical solution of a one-dimensional time fractional-order cancer model using the homotopy perturbation method (HPM). Initially, the fractional derivative component which is in the Caputo sense converted into an integer order derivative by using the Laplace transform method, followed by the technique mentioned above. The tasted numerical examples illustrate the feasibility and reliability of the proposed approach with a fractional-order derivative. Additionally, the impact of the fractional order on the solution's nature is analyzed graphically and numerically.

1. INTRODUCTION

Fractional calculus has gained much attention and is applied in numerous fields in engineering, viscoelasticity, fluid mechanics, electromagnetic waves, diffusion waves, earthquake, epidemiology, and mathematical biology. These all can be expressed in fractional order. Many mathematical models are partial differential equations (PDEs), making their explicit solution challenging. As a result, various ways to efficiently solve the models have emerged. For example, the fractional diffusion-wave equation is solved with the Riemann-Liouville fractional derivative. The method's convergence, stability, and consistency are demonstrated through both theoretical and numerical examples [1]. Similarly, finite difference approaches are utilized in this study to address the difficult two-dimensional cable problem [2]. In addition, the higher-order implicit finite difference iterative technique is proposed for solving two-dimensional temporal fractional Cable equations. Numerical examples and theoretical analysis demonstrated the method's utility [3]. Another study proposed a unique numerical approach (MFEGM) to solve the time-fractional advection-diffusion-reaction problem. Furthermore, this indicates that MFEGM outperforms the Crank-Nicolson finite difference [4]. Similarly, Khan et al. [5] find solitary wave solutions for the generalized Burgers-Huxley equation (B-HE) by solving partial differential equations (PDEs). Another study proposes a new auxiliary approach for the traveling wave solutions of the space-time fractional Cahn-Hilliard equation and the space-time fractional symmetric regularized long-wave equation [6]. Furthermore, in the article [7], the Khater approach provides new accurate traveling wave solutions to the non-linear space-time variable-order fractional shallow water wave equation in the Caputo fractional derivative sense. Moreover, Suardi et al. [8] focus on precious metal recycling and image processing of batteries to detect them in the e-waste to tackle the e-waste issues. The paper underlines a modified U-Net CNN with preprocessing, ensuring the accurate segmentation of batteries in X-ray images. This work is done as part of an effort to make e-waste recycling more sustainable, and according to the research, the most efficient one is the U-Net with a dice coefficient. Likewise, another numerical approach using the finite difference method is developed, and implicit algorithms such as Crank-Nicholson guarantee scalability and stability for long-term heat and mass transport simulations [9]. The optimal homotopy asymptotic method (OHAM) is utilized for the approximation of Lane-Emden and Emden-Fowler starting and boundary value problems by Khan et al. [10]. The obtained results are matched with HPM, ADM, and VIM. In this paper, the OHAM is used to approximate some well-known linear and nonlinear two-point boundary value problems. Results are compared with an exact solution and homotopy perturbation [11]. The unconditional stable method is proposed

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to solve the two-dimensional fractional Rayleigh-Stokes equation. This research investigated the stability, convergence, and accuracy analyses of the method combined with fractional derivatives and the fourth-order compact procedure [12]. In addition, an explicit group iterative method is proposed to approximate the solution of the two-dimensional fractional Rayleigh-Stokes problem for a heated generalized second-grade fluid. The numerical examples are used to verify the proposed approach, and the matrix method using a high-order compact C-N finite difference method is utilized to analyze the stability and convergence of the model [13]. The high-order compact iterative method is proposed for solving the two-dimensional time-fractional sub-diffusion equation. In the proposed approach, numerical examples verified the high-order convergence and effectiveness of the proposed method [14]. Similarly, a high-order implicit scheme method is presented for solving the two-dimensional time-fractional diffusion equation [15]. In the proposed method, convergence and accuracy are assessed using numerical examples. The closed-form traveling wave solution for the nonlinear fractional and variable-order fractional differential equations is discussed in [16, 17]. The reported results are based on the proposed method, which is very effective and valid for the models arising in mathematical physics. Salama et al. [18] considered a new modified hybrid explicit group iterative method to solve the two-space-dimension time-fractional diffusion equation. The approach is used as Laplace transformation for fractional operators and finite difference scheme-based group strategy for space derivatives. Theoretical analysis, numerical results, and comparison are carried out successfully, showing that the considered method is very accurate and efficient. In another study [19], they discussed the two explicit group numerical schemes through Crank-Nicolson schemes on different grids. The numerical results show that the said schemes reduced the computational time, number of iterations, and obtained high accuracy. Further, the comprehensive numerical study of the fractional-order differential and variable-order fractional equations can be seen in [20-35].

The above-discussed numerical approaches are feasible only for linear fractional differential equations. The nonlinear fractional differential equations can be solved by the approximate analytical approach and many researchers have solved complex nonlinear differential equations by various techniques such as Sadia et al. [36] applied a multi-step generalized differential transform method (MSGDTM) to the nonlinear model of the growth of tumor cells. They show that the solution acquired by MSGDTM is highly effective and accurate. Najafi and Basirzadeh [37] discussed optimal control HPM by using the HPM. They proved the effectiveness of OCHPM by comparing the attained numerical solution with HPM and claimed that the OCHPM method is effectual and originates powerful solutions for therapeutic models. Panchal and Patel [38] implemented the differential transform method (DTM) on the growth of the tumor model. They explained the outcomes of chemotherapy and revealed that if the concentration of chemotherapy drug is inappropriate then the growth of tumor cells increases in a large number or maybe causes a decrease in effector cells. Farman et al. [39] focused on a model of fractional-order immunotherapy bladder cancer and considered the BCG vaccine for its treatment where the derivative is described in Caputo of order $(0,1]$**.** They observed two cases of the growth rate of cells and claimed that both cases are stable in a fractional-order system. The result indicates that the fractional order provides great changes as compared to the classical derivative in terms of control of disease in the early stage. Kapoor [40] presented the solution of coupled 1D non-linear Burgers' equation by using HPM. The analytical solution is easily acquired by utilizing a general formula which is in the form of a recurrence relation. The exact solution is achieved in terms of power series (convergent in nature). Jitendra et al. [41] analyzed a mathematical model of a space-time fractional bio-heat equation. They used the fractional backward finite difference technique. They first converted the problem into an initial value problem and then applied HPM. Biazar and Aminikhah [42] adopted the variation iteration method (VIM) to work on a nonlinear Bergurs equation. By using VIM, it is feasible to unfold the exact solution as it is a highly effective method for solving a wide variety of problems. Compared to Adomian's decomposition method, VIM gives significant and efficient outcomes.

This study aims to find the approximate analytical solution for the time fractional-order cancer model. The Laplace transformation is used to convert the fractional order derivative operator into the order of an integer reducing the computational complexity. Further, the HPM method was applied to find the solution to the mentioned model. The approximate analytical solutions confirmed the implementation applicability and efficiency of the method. It shows that the proposed transformation reduces the complexity and computational work which is a more reliable and efficient tool for simulating the fractional order differential model. To the best of our knowledge, there is no similar work in the literature for the fractional-order cancer model.

The sequence of this paper is organized as follows: The literature is discussed in Section 1, and the basic definitions and properties are described in Section 2. In Section 3, The methodology of HPM is explained in detail. Section 4, explains the application and results, Section 5 adds the discussion part of the obtained graphical and numerical results, and the Conclusion is presented in Section 6.

In this study, consider the one-dimensional non-homogenous fractional-order cancer model [43]:

$$ {}^{c}D_t^{\alpha} f(x,t) = \frac{\partial^2 f(x,t)}{\partial x^2} - k(x,t) f(x,t), \quad t > 0, \; 0 < \alpha \le 1, \tag{1} $$

where ${}^{c}D_t^{\alpha}$ is a fractional-order differential operator which is defined in the Caputo sense, $k(x,t)$ symbolize the therapy-dependent killing ratio, whereas $f(x,t)$ indicate the number of tumor cells at position x and t represent the time. k cannot be specified, it can be expressed in three different ways, as a constant, a function of time, or a function that is not time-dependent.

2. Basic Preliminaries

This section presents the transformation for fractional order operator and basic definitions of fractional-order operator along with its properties as follows.

Definition 1. The time-fractional Caputo derivative operator with order α of the function $f(x)$ is defined as [37].

$$D_*^\alpha f(x) = J^{m-\alpha} D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^\alpha (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad (2)$$

for $m-1 < \alpha \le m, m \in Z, x > 0, f \in C_{-1}^m$.

Some properties are as follows:

If $m-1 < \alpha \le m, m \in Z, x > 0, f \in C_\mu^m, \mu \ge -1$, then

$D_*^\alpha J^\alpha f(x) = f(x)$,

$J^\alpha D_*^\alpha f(x) = f(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)x^k}{k!}, \quad x > 0.$

Definition 2. The transformed form of fractional order Caputo derivative operator of order α can be defined as

$${}^cD_t^\alpha f(x,t) = \alpha \frac{\partial f}{\partial t} + (1-\alpha) f(x,t) - (1-\alpha) f(x,0). \quad (3)$$

3. Homotopy Perturbation Method (HPM)

The homotopy perturbation method (HPM) is a semi-analytical technique for solving linear as well as nonlinear fractional-order partial differential equations. The method may also be used to solve a system of coupled linear and nonlinear differential equations.

Consider the general form of the following differential equation

$${}^cD_t^\alpha(f) + L(f) + R(f) + N(f) = 0,\ m-1 < \alpha < m, \quad (4)$$

we can assume the solution of the equation in the form

$$f = \sum_{k=0}^\infty f_k.$$

Applying the proposed technique in Eq (4)

$$\frac{\partial^m f(x,t)}{\partial t^m} - f^*(x,t) = p\left[\frac{\partial^m f}{\partial t^m} - {}^cD_t^\alpha(f) - L(f) - R(f) - N(f)\right].$$

To perturb Eq (11), put $m = 1$ for the case of $0 < \alpha < 1$ and assume that

$$f = u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 + p^4 u_4 + \cdots$$

$$\frac{\partial}{\partial t}(u_0 + p^1 u_1 + \cdots) - f^*(x,t) = p\left[\frac{\partial}{\partial t}(u_0 + p^1 u_1 + \cdots) - {}^cD_t^\alpha(u_0 + p^1 u_1 + \cdots) - L(u_0 + p^1 u_1 + \cdots) - R(u_0 + p^1 u_1 + \cdots) - N(u_0 + p^1 u_1 + \cdots)\right], \quad (5)$$

where $f^*(x,t)$ is the forcing term and equating the coefficients of like powers of p, we get

$$p^0: \quad \frac{\partial}{\partial t} u_0(x,t) = f^*(x,t),$$

$$p^1: \quad \frac{\partial}{\partial t} u_1 = \left[\frac{\partial}{\partial t} u_0 - {}^cD_t^\alpha u_0 - Lu_0 - Ru_0 - Nu_0\right],$$

$$p^2: \quad \frac{\partial}{\partial t} u_2 = \left[\frac{\partial}{\partial t} u_1 - {}^cD_t^\alpha u_1 - Lu_1 - Ru_1 - Nu_1\right],$$

$$\vdots$$

$$p^{k+1}: \quad \frac{\partial}{\partial t} u_{k+1} = \left[\frac{\partial}{\partial t} u_k - {}^cD_t^\alpha u_k - Lu_k - Ru_k - Nu_k\right], k \ge 1. \quad (6)$$

As p varies from 0 to 1, the solution changes from the initial value $u_0(x,t)$ to the solution $u(x,t)$, in topology this variation is called deformation or homotopy. Here, consider the solution of Eq(5) in the form of power series in p is defined as

$$u(x,t) = \sum_{k=0}^{\infty} u_k\, p^k.$$

The approximate solution of Eq(1) can also be defined as:

$$u(x,t) = \lim_{p\to 1} u = \sum_{k=0}^{\infty} u_k. \tag{7}$$

4. Numerical Applications

This section discusses the numerical applications of the proposed model and implements the HPM method to find out the approximate solution and graphical representation for the proposed fractional-order cancer equation [43, 44], by using the Maple 15 software.

Example 1. Consider the time fractional-order equation of the clear killing ratio of the cancer cells.

$${}^{c}D_t^{\alpha}\Psi(x,t) = \frac{\partial^2}{\partial x^2}\Psi(x,t) - t^2\Psi(x,t),\ t>0,\ 0\le x\le 1,\ 0<\alpha\le 1. \tag{8}$$

Subject to the initial condition

$$\Psi(x,0) = e^{rx}. \tag{9}$$

Using definition 2 for fractional-order derivative, we obtained

$$\frac{\partial\Psi(x,t)}{\partial t} + \frac{(1-\alpha)}{\alpha}\big(\Psi(x,t)-\Psi(x,0)\big) - \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t) + \frac{t^2}{\alpha}\Psi(x,t), \tag{10}$$

now, when applying for the HPM, we get

$$\frac{\partial^m\Psi(x,t)}{\partial t^m} - \Psi^*(x,t) = p\left[\frac{\partial^m\Psi}{\partial t^m} - \frac{\partial\Psi}{\partial t} - \left(\frac{1-\alpha}{\alpha}\right)\big(\Psi(x,t)-\Psi(x,0)\big) + \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t) - \frac{t^2}{\alpha}\Psi(x,t)\right],$$

where $\Psi^*(x,t)$ is the forcing term, and for $m=1$, we get the following form

$$\frac{\partial\Psi(x,t)}{\partial t} = p\left[-\left(\frac{1-\alpha}{\alpha}\right)\,[\Psi(x,t)-\Psi(x,0)] + \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t) - \frac{t^2}{\alpha}\Psi(x,t)\right]. \tag{11}$$

To perturb Eq (11), Assuming that

$$\Psi = \sum_{n=0}^{\infty} p^n\Psi_n,\ \text{ or }\ \Psi = \Psi_0 + p^1\Psi_1 + p^2\Psi_2 + \cdots,$$

hence putting Eq (12) into Eq(11) and equating the like power of p, we obtained the following recurrence relation.

$$\Psi_0(x,t) = \mathrm{e}^{rx},$$

$$\frac{\partial}{\partial t}\Psi_{k+1}(x,t) = -\left(\frac{1-\alpha}{\alpha}\right)\,[\Psi_k(x,t)-\Psi_k(x,0)] + \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi_k(x,t) - \frac{t^2}{\alpha}\Psi_k(x,t).\ k\ge 0. \tag{12}$$

Consequently, we have the following components of the solution.

$$\Psi_0(x,t) = \mathrm{e}^{rx},$$

$$\Psi_1(x,t) = \frac{\mathrm{e}^{rx}r^2t}{\alpha} - \frac{\mathrm{e}^{rx}t^3}{3\alpha},$$

$$\Psi_2(x,t) = \frac{\mathrm{e}^{rx}t^2\big(18r^4 - 6r^2(3+2t^2-3\alpha) + t^2(3+2t^2-3\alpha)\big)}{36\alpha^2},$$

$$\Psi_3(x,t) = -\frac{1}{36\alpha^3}\mathrm{e}^{rx}\left(\frac{2t^9}{9}+\frac{3}{5}t^5\left(10r^4+11r^2(-1+\alpha)+(-1+\alpha)^2\right)-6r^2t^3(-1+r^2+\alpha)^2-\frac{1}{7}t^7(-5+14r^2+5\alpha)\right),$$

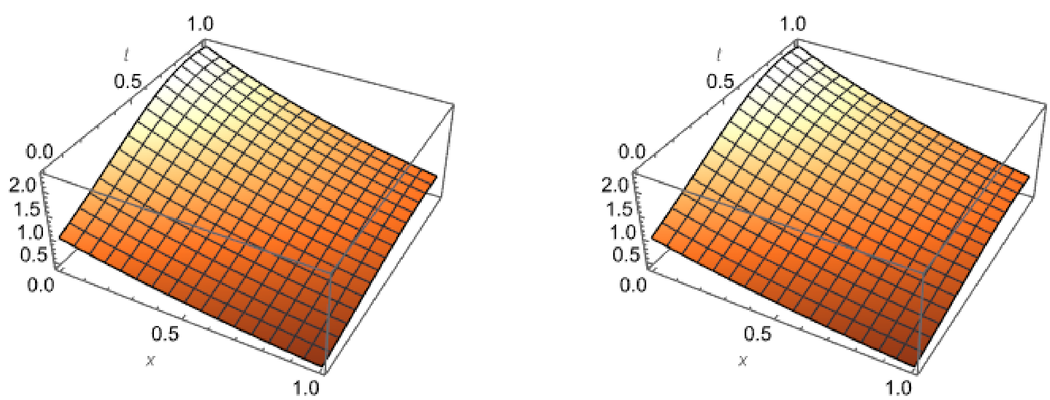
$$\Psi_4(x,t) = \frac{1}{136080\alpha^4}\mathrm{e}^{rx}t^4\left(70t^8-6t^6\big(140r^2+59(-1+\alpha)\big)+27t^4\left(140r^4+163r^2(-1+\alpha)+23(-1+\alpha)^2\right)+5670r^2(-1+r^2+\alpha)^3-378t^2(-1+r^2+\alpha)^2(-1+20r^2+\alpha)\right).$$

$$\vdots$$

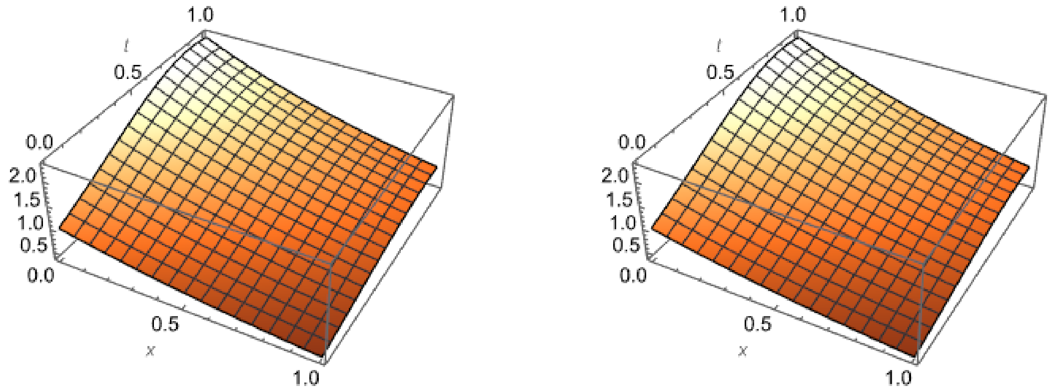
Hence, the solution of equations (8) and (9) can be expressed in the series expansion

$$\Psi(x,t) = \frac{1}{136080\alpha^5}\mathrm{e}^{rx}\Bigg(-\frac{14t^{15}}{3}+\frac{2}{13}t^{13}\big(455r^2+212(-1+\alpha)\big)-\frac{15}{11}t^{11}\left(308r^4+373r^2(-1+\alpha)+65(-1+\alpha)^2\right)+3780t^2\big(18r^4-6r^2(3+2t^2-3\alpha)+t^2(3+2t^2-3\alpha)\big)\alpha^3+136080r^2t\alpha^4-45360t^3\alpha^4+136080\alpha^5+3t^9\big(420r^2+37(-1+\alpha)\big)(-1+r^2+\alpha)^2+1134r^2t^5(-1+r^2+\alpha)^4-54t^7(-1+r^2+\alpha)^3(-1+35r^2+\alpha)+t^4\alpha\left(70t^8-6t^6\big(140r^2+59(-1+\alpha)\big)+27t^4\left(140r^4+163r^2(-1+\alpha)+23(-1+\alpha)^2\right)+5670r^2(-1+r^2+\alpha)^3-378t^2(-1+r^2+\alpha)^2(-1+20r^2+\alpha)\right)-3780\alpha^2\left(\frac{2t^9}{9}+\frac{3}{5}t^5\left(10r^4+11r^2(-1+\alpha)+(-1+\alpha)^2\right)-6r^2t^3(-1+r^2+\alpha)^2-\frac{1}{7}t^7(-5+14r^2+5\alpha)\right)\Bigg).$$

Moreover, the contour graphs and 3D graphs of the fifth iterative series expansion $\Psi(x,t)$ of example 1 are presented below.



(a). $\alpha = 1$ **(b).** $\alpha = 0.9$



(c). $\alpha = 0.8$ **(d).** $\alpha = 0.6$

Fig. 1. The 3D graph of the solution $\Psi(x,t)$ of the fractional order of cancer tumor model at $r = -1$ for different values of fractional-order α.

TABLE I. NUMERICAL RESULT OF THE EXACT AND APPROXIMATE SOLUTIONS

x	Exact Solution	Approximate Solution	Error
1.0e-01	0.951200634900000	0.951189788600000	1.08463E-05
2.0e-01	0.860681926600000	0.860672112500000	9.81410E-06
3.0e-01	0.778777212200000	0.778768332000000	8.88020E-06
4.0e-01	0.704666761900000	0.704658726700000	8.03520E-06
5.0e-01	0.637608853400000	0.637601582900000	7.27050E-06
6.0e-01	0.576932348600000	0.576925770000000	6.57860E-06
7.0e-01	0.522029976700000	0.522024024200000	5.95250E-06
8.0e-01	0.472352256300000	0.472346870100000	5.38620E-06
9.0e-01	0.427401995900000	0.427397122400000	4.87350E-06

Example 2. Consider the time fractional equation of the clear killing ratio of the cancer cells

$$D_t^{\alpha}\Psi(x,t) = \frac{\partial^2}{\partial x^2}\Psi(x,t) - \frac{2}{x^2}\Psi(x,t), \quad t > 0, \ \ 0 \le x \le 1, \ \ 0 < \alpha \le 1. \tag{13}$$

Subject to the initial condition

$$\Psi(x,0) = \frac{1}{x} + x^2. \tag{14}$$

Using definition 2 for fractional-order derivative, we obtained

$$\frac{\partial\Psi(x,t)}{\partial t} + \frac{(1-\alpha)}{\alpha}\big(\Psi(x,t) - \Psi(x,0)\big) - \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t) + \frac{2}{\alpha x^2}\Psi(x,t),$$

now applying for the HPM, we get

$$\frac{\partial^m\Psi(x,t)}{\partial t^m} - \Psi^*(x,t) = p\left[\frac{\partial^m\Psi}{\partial t^m} - \frac{\partial\Psi}{\partial t} - \left(\frac{1-\alpha}{\alpha}\right)\big(\Psi(x,t) - \Psi(x,0)\big) + \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t) - \frac{2}{\alpha x^2}\Psi(x,t)\right],$$

where $\Psi^*(x,t)$ is the forcing term, and for $m = 1$, we get the following form

$$\frac{\partial\Psi(x,t)}{\partial t} = p\left[-\left(\frac{1-\alpha}{\alpha}\right)[\Psi(x,t) - \Psi(x,0)] + \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t) - \frac{2}{\alpha x^2}\Psi(x,t)\right]. \tag{15}$$

To perturb the above equation, Assuming that

$$\Psi = \sum_{n=0}^{\infty} p^n \Psi_n, \text{ ορ } \Psi = \Psi_0 + p^1\Psi_1 + p^2\Psi_2 + \cdots, \tag{16}$$

hence putting Eq (16) into Eq(15) and equating the like power of p, we obtained the following recurrence relation.

$$\Psi_0(x,t) = e^{rx},$$

$$\frac{\partial}{\partial t}\Psi_{k+1}(x,t) = -\left(\frac{1-\alpha}{\alpha}\right)[\Psi_k(x,t) - \Psi_k(x,0)] + \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi_k(x,t) - \frac{2}{\alpha x^2}\Psi_k(x,t).\, k \geq 0. \tag{17}$$

Consequently, we have the following components of the solution.

$$\Psi_0(x,t) = \frac{1}{x} + x^2,$$

$$\Psi_1(x,t) = \frac{t\left(2+\frac{2}{x^3}\right)}{\alpha} - \frac{2t\left(\frac{1}{x}+x^2\right)}{x^2\alpha},$$

$$\Psi_2(x,t) = 0,$$

$$\Psi_3(x,t) = 0.$$

Hence the series solution of equations (13) and (14) would be

$$\Psi(x,t) = \frac{1}{x} + x^2.$$

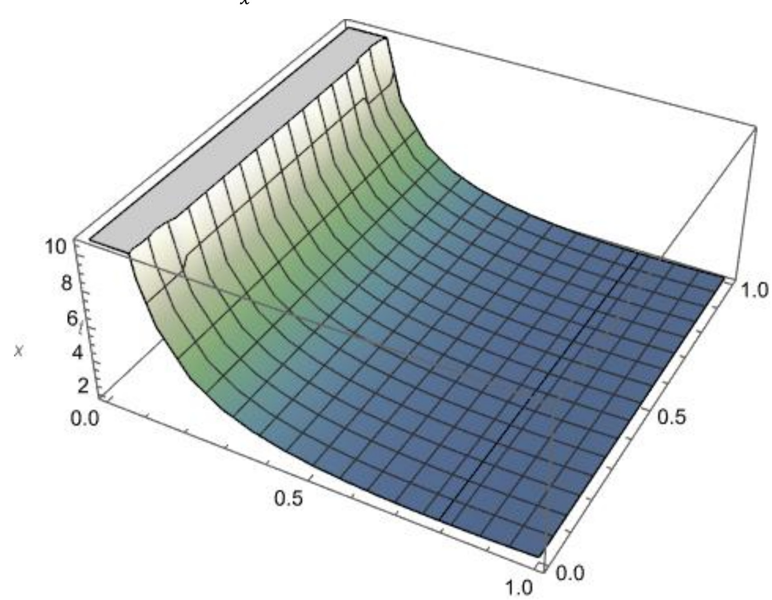


Fig. 2. Graph of the solution $\Psi(x,t)$ of fractional order cancer tumor model at $\alpha = 1$ at $r = -1$.

Example 3. Consider the nonlinear time-fractional equation of the clear killing ratio of the cancer cells

$$D_t^{\alpha}\Psi(x,t) = \frac{\partial^2}{\partial x^2}\Psi(x,t) - \frac{2}{x}\frac{\partial}{\partial x}\Psi(x,t) - \Psi^2(x,t), 0 < \alpha \leq 1, t > 0, 0 \leq x \leq 1. \tag{18}$$

Subject to the initial condition:

$$\Psi(x,0) = x^p, \quad p > 0. \tag{19}$$

Using definition 2 for fractional-order derivative, we obtained

$$\frac{\partial\Psi(x,t)}{\partial t}+\frac{(1-\alpha)}{\alpha}\left(\Psi(x,t)-\Psi(x,0)\right)-\frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t)-\frac{2}{\alpha x}\frac{\partial}{\partial x}\Psi(x,t)-\frac{1}{\alpha}\Psi^2(x,t),$$

now applying the HPM, we get

$$\frac{\partial^m\Psi(x,t)}{\partial t^m}-\Psi^*(x,t)=p\left[\frac{\partial^m\Psi}{\partial t^m}-\frac{\partial\Psi}{\partial t}-\left(\frac{1-\alpha}{\alpha}\right)\left(\Psi(x,t)-\Psi(x,0)\right)+\frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t)-\frac{2}{\alpha x}\frac{\partial}{\partial x}\Psi(x,t)-\frac{1}{\alpha}\Psi^2(x,t)\right],$$

where $\Psi^*(x,t)$ is the forcing term, and for $m=1$, we get the following form

$$\frac{\partial\Psi(x,t)}{\partial t}=p\left[-\left(\frac{1-\alpha}{\alpha}\right)\left[\Psi(x,t)-\Psi(x,0)\right]+\frac{1}{\alpha}\frac{\partial^2}{\partial x^2}\Psi(x,t)-\frac{2}{\alpha x}\frac{\partial}{\partial x}\Psi(x,t)-\frac{1}{\alpha}\Psi^2(x,t)\right]. \quad (20)$$

To perturb the above equation, Assuming that

$$\Psi=\sum_{n=0}^{\infty}p^n\Psi_n,\ \text{oρ}\ \Psi=\Psi_0+p^1\Psi_1+p^2\Psi_2+\cdots \quad (21)$$

hence putting Eq (21) into Eq(20) and equating the like power of p, we obtained the following recurrence relation.

$$\Psi_0(x,t)=x^p,$$

$$\Psi_1(x,t)=\frac{tx^{-2+p}(-3p+p^2-x^{2+p})}{\alpha},$$

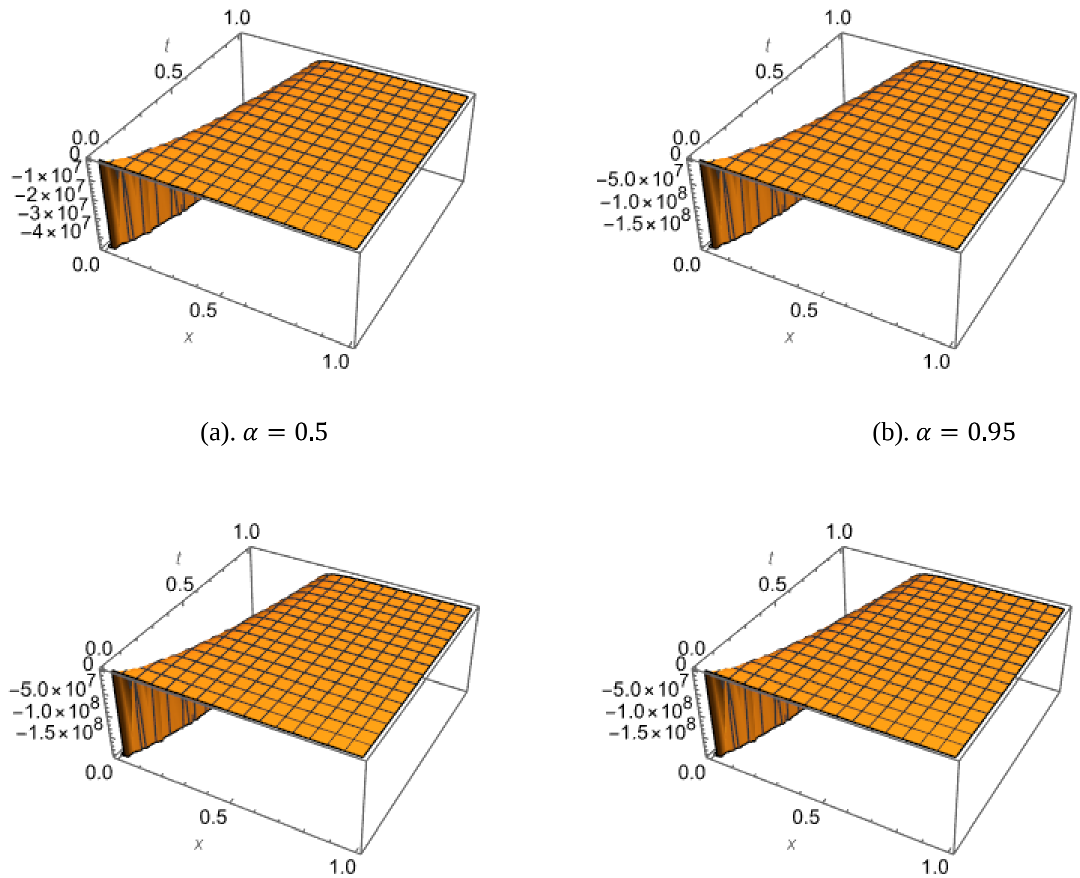
$$\Psi_2(x,t)=-\frac{1}{6\alpha^3}t^2x^{-4+p}\Big(-6p^3(2tx^p-5\alpha)+p^4(2tx^p-3\alpha)+p^2\big(-2tx^p(-9+2x^{2+p})-3\big(31-4x^{2+p}+x^2(-1+\alpha)\big)\alpha\big)+3p\big(4tx^{2+2p}+3\big(10-2x^{2+p}+x^2(-1+\alpha)\big)\alpha\big)+x^{4+p}(2tx^{2p}+3(-1+\alpha)\alpha)\Big),$$

$$\Psi_3(x,t)=\frac{1}{6\alpha^4}x^{-6+p}\Big(\frac{1}{3}t^6x^{2+p}(3p-p^2+x^{2+p})^2-\frac{1}{2}t^4x^p\Big(-42p^5+4p^6-2px^{2+p}\big(-18+2x^{2+p}-3x^2(-1+\alpha)\big)+6p^3\big(-47+14x^{2+p}-x^2(-1+\alpha)\big)+p^4\big(164-18x^{2+p}+x^2(-1+\alpha)\big)+p^2\big(180-102x^{2+p}+16x^{4+2p}+9x^2(-1+\alpha)-2x^{4+p}(-1+\alpha)\big)+x^{6+2p}(-1+\alpha)\Big)+\frac{3}{5}t^5x^2\Big(10p^3-p^4+p^2\big(-31+4x^{2+p}-x^2(-1+\alpha)\big)+3p\big(10-2x^{2+p}+x^2(-1+\alpha)\big)+x^{4+p}(-1+\alpha)\Big)\alpha+t^3\big(-19p^5+p^6+p^3\big(-509+64x^{2+p}-18x^2(-1+\alpha)\big)+p^4\big(141-16x^{2+p}+2x^2(-1+\alpha)\big)+p(-600+36x^{2+p}-48x^2(-1+\alpha)+8x^{4+p}(-1+\alpha)-3x^4(-1+\alpha)^2)+p^2(890-84x^{2+p}+52x^2(-1+\alpha)-8x^{4+p}(-1+\alpha)+x^4(-1+\alpha)^2)-x^{6+p}(-1+\alpha)^2)\alpha\Big).$$

Hence the series expansion is

$$\Psi(x,t)=x^p+\frac{tx^{-2+p}(-3p+p^2-x^{2+p})}{\alpha}+\frac{1}{6\alpha^4}x^{-6+p}\Big(\frac{1}{3}t^6x^{2+p}(3p-p^2+x^{2+p})^2-\frac{1}{2}t^4x^p\Big(-42p^5+4p^6-2px^{2+p}\big(-18+2x^{2+p}-3x^2(-1+\alpha)\big)+6p^3\big(-47+14x^{2+p}-x^2(-1+\alpha)\big)+p^4\big(164-18x^{2+p}+x^2(-1+\alpha)\big)+p^2\big(180-102x^{2+p}+16x^{4+2p}+9x^2(-1+\alpha)-2x^{4+p}(-1+\alpha)\big)+x^{6+2p}(-1+\alpha)\Big)+\frac{3}{5}t^5x^2\Big(10p^3-p^4+p^2\big(-31+4x^{2+p}-x^2(-1+\alpha)\big)+3p\big(10-2x^{2+p}+x^2(-1+\alpha)\big)+x^{4+p}(-1+\alpha)\Big)\alpha+t^3\big(-19p^5+p^6+p^3\big(-509+64x^{2+p}-18x^2(-1+\alpha)\big)+p^4\big(141-16x^{2+p}+2x^2(-1+\alpha)\big)+p(-600+36x^{2+p}-48x^2(-1+\alpha)+8x^{4+p}(-1+\alpha)-3x^4(-1+\alpha)^2)+p^2(890-84x^{2+p}+52x^2(-1+\alpha)-8x^{4+p}(-1+\alpha)+x^4(-1+\alpha)^2)-x^{6+p}(-1+\alpha)^2)\alpha\Big)-\frac{1}{180\alpha^5}x^{-8+p}\Big(\frac{10}{9}t^9x^{4+p}(3p-p^2+x^{2+p})^2-\frac{25}{7}t^7x^{2+p}+\cdots.$$

Moreover, the contour graphs and 3D graphs of the fifth iterative series expansion $\Psi(x,t)$ of example 3 are presented below.



(a). $\alpha = 0.5$ (b). $\alpha = 0.95$

(c). $\alpha = 0.85$ (d). $\alpha = 0.75$

Fig. 3. The 3D graph of the solution $\Psi(x,t)$ of the fractional order cancer tumor model at $p = 1.2, r = -1$ for different values of α.

5. Discussion

This section presents some graphical solutions of the proposed fractional order cancer model solved by the HPM method. In Figure 1, the 3D graphs are plotted for example 1 for the solution $\Psi(x,t)$ of fractional order cancer tumor model at $r = -1$ for different values of fractional-order $\alpha = 0.6, 0.8, 0.9$, and $\alpha = 1$, to examine the effects of α on the tumor cells and also the numerical results in table 1 show excellent agreement with the exact solution. Figure 2 represents the 3D plot for example 2 which coincides with the graph obtained in [34]. It shows that the concentricity of cancer tumor cells reduces over time and finally approaches zero. Figure 3 is plotted for example 3, which is the graphical representation for different values of $\alpha = 0.5, 0.75, 0.85, 0.95, p = 1.2$, and $r = -1$. In all these cases, examined that with increasing time the concentricity of the cancer cells decreases and finally approaches zero.

6. Conclusion

This article analyzed the number of tumor cells as a function of fractional-order derivative and implemented the HPM with Laplace transformation as it provides a very powerful and efficient approximate solution for the fractional-order cancer models. The result we achieved in terms of series which is easy to deal with. The 3D Graphical representation and numerical error in Table 1 also supported our numerical results. Moreover, the outcome based on transformation indicates that this method is simple, efficient, and highly powerful, and this transformation can be extended to other types of fractional higher-order physical and biological models.

Conflicts of Interest

The authors declare no conflicts of interest.

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