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Review Article

A Comprehensive Review Note on Pachpatte-Type Integral Inequality

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ABSTRACT

Convexity is mainly utilized to describe a number of problems in both applied and pure mathematics. The main focus and objective of this review paper is to present the Pachpatte-type inequalities via different type of convexities.

1. INTRODUCTION

Convex functions have a long and illustrious history. The history of convexity theory can be traced all the way back to the end of the nineteenth century. The word "convex functions" has been widely used and explored in the well-known and popular book namely "Inequalities," published by G. Polya, G.H. Hardy, and J.E. Littlewood [1]. This book is quickly became a standard reference for mathematicians and dedicated solely to the topic of inequality and serves as an excellent introduction to this fascinating field. Convex theory provides us with appropriate guidelines and techniques to focus on a broad range of problems in applied sciences. It has been widely acknowledged in recent years that mathematical inequalities have contributed to the development of various aspects of mathematics as well as other scientific disciplines. This theory have remarkable uses in economics [2], optimization [3], mathematical optimization for modeling [4, 5], finance [6], computer science [7], control systems [8] and estimation and signal processing [9]. This theory provides a solid framework for the start and growth of numerical tools for the study and solution of challenging mathematical issues. The investigation of inequality has been regarded and viewed as one of the key fields of applied sciences. It is a fast-growing science with a rising number of applications in many scientific domains. Integral inequalities have fruitful importance in integral operator theory, stochastic processes, numerical integration, optimization theory, probability, statistics and information technology.

Within the realm of mathematical interpretation, the combined study of integral inequalities and convex analysis offers an intriguing and compelling topic of study. The strategies and literature of convex analysis and integral inequalities have recently become the focus of intense research in both the present and the past due to their broad viewpoints and applications.

In this study, we aim to provide an in-depth and current review of Pachpatte-type inequalities for various convexity types. We think that by compiling nearly all of the Pachpatte-type inequalities that have been published in the literature into

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a single file, aspiring researchers will be better informed about the body of work on the subject before coming up with their own findings. For the reader's convenience, we include a comprehensive reference for every outcome detailed in this survey rather than presenting the results without supporting evidence.
Due to its numerous applications in physics and mathematics, mathematical inequalities are essential to the study of mathematics as well as other branches of the subject. The convex function, which has the following definition, is one of the most important functions used to investigate a variety of intriguing inequalities:
Let $\mathbb{I} \subset \mathbb{R}$ be a non-empty interval. The function $\Pi : \mathbb{I} \to \mathbb{R}$ is called convex if

$$\Pi(\lambda\delta_1 + (1-\lambda)\delta_2) \leq \lambda\Pi(\delta_1) + (1-\lambda)\Pi(\delta_2),$$

holds for every $\delta_1, \delta_2 \in \mathbb{I}$ and $\lambda \in [0,1]$.
Currently, many researchers have been fascinated by the field of convex functions and, particularly, one of the well-known inequalities for convex functions known as the Pachpatte-type inequality, which is defined as follows: Pachpatte type inequality [10] asserts that:

Theorem 1.1. *[10] If Π_1 and Π_2 be a non-negative real-valued convex function on $[\delta_1, \delta_2]$, then*

$$\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Pi_1(\wp)\Pi_2(\wp)d\wp \leq \frac{1}{3}\mathtt{M}(\delta_1,\delta_2) + \frac{1}{6}\mathtt{N}(\delta_1,\delta_2),$$

and

$$2\Pi_1\left(\frac{\delta_1+\delta_2}{2}\right)\Pi_2\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Pi_1(\wp)\Pi_2(\wp)d\wp + \frac{1}{3}\mathtt{M}(\delta_1,\delta_2) + \frac{1}{6}\mathtt{N}(\delta_1,\delta_2),$$

where $\mathtt{M}(\delta_1,\delta_2) = \Pi_1(\delta_1)\Pi_2(\delta_1) + \Pi_1(\delta_2)\Pi_2(\delta_2)$, $\mathtt{N}(\delta_1,\delta_2) = \Pi_1(\delta_1)\Pi_2(\delta_2) + \Pi_1(\delta_2)\Pi_2(\delta_1)$.

In 2007, Kirmaci et.al [11] introduced the new variant of Pachpatte-type inequalities via s-convex functions in classical calculus as follows:

Theorem 1.2. *[11] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R, \delta_1, \delta_2 \in [0,\infty)$, $\delta_1 < \delta_2$ be functions such that $\Pi_2, \Pi_1\Pi_2 \in L[\delta_1,\delta_2]$. If Π_1 is convex and non-negative on $[\delta_1,\delta_2]$ and Π_2 is s-convex on $[\delta_1,\delta_2]$ for some fixed $s \in (0,1]$, then*

$$\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Pi_1(x)\Pi_2(x)dx \leq \frac{1}{s+2}\mathtt{M}(\delta_1,\delta_2) + \frac{1}{(s+1)(s+2)}\mathtt{N}(\delta_1,\delta_2),$$

where $\mathtt{M}(\delta_1,\delta_2)$ and $\mathtt{N}(\delta_1,\delta_2)$ are defined in Theorem 1.1.

Theorem 1.3. *[11] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R, \delta_1, \delta_2 \in [0,\infty)$, $\delta_1 < \delta_2$ be functions such that Π_1, Π_2 and $\Pi_1\Pi_2 \in L[\delta_1,\delta_2]$. If Π_1 is s_1-convex on $[\delta_1,\delta_2]$ and Π_2 is s_2-convex on $[\delta_1,\delta_2]$ for some fixed $s_1, s_1 \in (0,1]$, then*

$$\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Pi_1(x)\Pi_2(x)dx \leq \frac{1}{s_1+s_2+1}\mathtt{M}(\delta_1,\delta_2) + \beta(s_2+1, s_1+1)\mathtt{N}(\delta_1,\delta_2),$$

where $\mathtt{M}(\delta_1,\delta_2)$ and $\mathtt{N}(\delta_1,\delta_2)$ are defined in Theorem 1.1.

Theorem 1.4. *[11] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R, \delta_1, \delta_2 \in [0,\infty)$, $\delta_1 < \delta_2$ be functions such that Π_1, Π_2 and $\Pi_1\Pi_2 \in L[\delta_1,\delta_2]$. If Π_1 is convex on $[\delta_1,\delta_2]$ and Π_2 is s-convex on $[\delta_1,\delta_2]$ for some fixed $s \in (0,1]$, then*

$$\begin{aligned}&2^s\Pi_1\left(\frac{\delta_1+\delta_2}{2}\right)\Pi_2\left(\frac{\delta_1+\delta_2}{2}\right)\\ &\leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Pi_1(x)\Pi_2(x)dx + \frac{1}{(s+1)(s+2)}\mathtt{M}(\delta_1,\delta_2) + \frac{1}{s+2}\mathtt{N}(\delta_1,\delta_2),\end{aligned}$$

where $\mathtt{M}(\delta_1,\delta_2)$ and $\mathtt{N}(\delta_1,\delta_2)$ are defined in Theorem 1.1.

In 2014, Chen et.al [12] introduced a new variant of Pachpatte-type inequalities via harmonically s-convex functions in classical calculus as follows:

Theorem 1.5. *[12] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R, \delta_1, \delta_2 \in [0,\infty)$, $\delta_1 < \delta_2$ be functions such that Π_1, Π_2 and $\Pi_1\Pi_2 \in L[\delta_1,\delta_2]$. If Π_1 is harmonically s_1-convex on $[\delta_1,\delta_2]$ and Π_2 is harmonically s_2-convex on $[\delta_1,\delta_2]$ for some fixed $s_1, s_1 \in (0,1]$, then*

$$\frac{\delta_1\delta_2}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \frac{\Pi_1(x)\Pi_2(x)}{x^2}dx \leq \frac{1}{1+s_1+s_2}\mathtt{M}(\delta_1,\delta_2) + \frac{\Gamma(1+s_1)\Gamma(1+s_2)}{\Gamma(s_1+s_2+2)}\mathtt{N}(\delta_1,\delta_2),$$

where $\mathtt{M}(\delta_1,\delta_2)$ and $\mathtt{N}(\delta_1,\delta_2)$ are defined in Theorem 1.1.

Theorem 1.6. *[12] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R, \delta_1, \delta_2 \in [0, \infty)$, $\delta_1 < \delta_2$ be functions such that Π_1, Π_2 and $\Pi_1\Pi_2 \in L[\delta_1, \delta_2]$. If Π_1 is harmonically s_1-convex on $[\delta_1, \delta_2]$ and Π_2 is harmonically s_2-convex on $[\delta_1, \delta_2]$ for some fixed $s_1, s_1 \in (0, 1]$, then*

$$\begin{aligned} &2^{s_1+s_2-1}\Pi_1\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right)\Pi_2\left(\frac{2\delta_1\delta_2}{\delta_1+\delta_2}\right) \\ &\leq \frac{\delta_1\delta_2}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\frac{\Pi_1(x)\Pi_2(x)}{x^2}dx + \mathtt{M}(\delta_1,\delta_2)\frac{\Gamma(1+s_1)\Gamma(1+s_2)}{\Gamma(s_1+s_2+2)} + \mathtt{N}(\delta_1,\delta_2)\frac{1}{1+s_1+s_2}, \end{aligned}$$

where $\mathtt{M}(\delta_1, \delta_2)$ and $\mathtt{N}(\delta_1, \delta_2)$ are defined in Theorem 1.1.

In 2022, Sahoo et.al [13] introduced the new variant of Pachpatte-type inequalities via CR-h-preinvex functions in interval analysis as follows:

Theorem 1.7. *[13] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$, is an interval functions and $\Pi_1, \Pi_2 \in IR_{([\delta_2,\delta_1])}$. If $\Pi_1 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$ is a CR-h_1-preinvex function and $\Pi_2 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$ is a CR-h_2-preinvex function, then the following inequalities hold true:*

$$\begin{aligned} &\frac{1}{\eta(\delta_2,\delta_1)}\int_{\delta_1}^{\delta_1+\eta(\delta_2,\delta_1)}\Pi_1(z)\Pi_2(z)dz \\ &\leq_{CR} \mathtt{M}(\delta_1,\delta_2)\int_0^1 h_1(1-t)h_2(1-t)dt + \mathtt{N}(\delta_1,\delta_2)\int_0^1 h_1(1-t)h_2(t)dt, \end{aligned}$$

where $\mathtt{M}(\delta_1, \delta_2)$ and $\mathtt{N}(\delta_1, \delta_2)$ are defined in Theorem 1.1.

Theorem 1.8. *[13] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$, is an interval functions and $\Pi_1, \Pi_2 \in IR_{([\delta_2,\delta_1])}$. If $\Pi_1 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$ is a CR-h_1-preinvex function and $\Pi_2 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$ is a CR-h_2-preinvex function, then the following inequalities hold true:*

$$\begin{aligned} &\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\Pi_1\left(\delta_1+\frac{1}{2}\eta(\delta_2,\delta_1)\right)\Pi_2\left(\delta_1+\frac{1}{2}\eta(\delta_2,\delta_1)\right) \\ &\leq_{CR} \frac{1}{\eta(\delta_2,\delta_1)}\int_{\delta_1}^{\delta_1+\eta(\delta_2,\delta_1)}\Pi_1(z)\Pi_2(z)dz \\ &+ \mathtt{M}(\delta_1,\delta_2)\int_0^1 h_1(1-t)h_2(t)dt + \mathtt{N}(\delta_1,\delta_2)\int_0^1 h_1(1-t)h_2(1-t)dt, \end{aligned}$$

where $\mathtt{M}(\delta_1, \delta_2)$ and $\mathtt{N}(\delta_1, \delta_2)$ are defined in Theorem 1.1.

In 2022, Sahoo et.al [13] introduced the new variant of Pachpatte-type inequalities via CR-h-convex functions in interval analysis as follows:

Theorem 1.9. *[13] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R$, is an interval functions and $\Pi_1, \Pi_2 \in IR_{([\delta_2,\delta_1])}$. If $\Pi_1 : [\delta_1, \delta_2] \to R$ is a CR-h_1-convex function and $\Pi_2 : [\delta_1, \delta_2] \to R$ is a CR-h_2-convex function, then the following inequalities hold true:*

$$\begin{aligned} &\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Pi_1(z)\Pi_2(z)dz \\ &\leq_{CR} \mathtt{M}(\delta_1,\delta_2)\int_0^1 h_1(1-t)h_2(1-t)dt + \mathtt{N}(\delta_1,\delta_2)\int_0^1 h_1(1-t)h_2(t)dt, \end{aligned}$$

where $\mathtt{M}(\delta_1, \delta_2)$ and $\mathtt{N}(\delta_1, \delta_2)$ are defined in Theorem 1.1.

Theorem 1.10. *[13] Let $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R$, is an interval functions and $\Pi_1, \Pi_2 \in IR_{([\delta_2,\delta_1])}$. If $\Pi_1 : [\delta_1, \delta_2] \to R$ is a CR-h_1-convex function and $\Pi_2 : [\delta_1, \delta_2] \to R$ is a CR-h_2-convex function, then the following inequalities hold true:*

$$\begin{aligned} &\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\Pi_1\left(\frac{\delta_1+\delta_2}{2}\right)\Pi_2\left(\frac{\delta_1+\delta_2}{2}\right) \\ &\leq_{CR} \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Pi_1(z)\Pi_2(z)dz \\ &+ \mathtt{M}(\delta_1,\delta_2)\int_0^1 h_1(1-t)h_2(t)dt + \mathtt{N}(\delta_1,\delta_2)\int_0^1 h_1(1-t)h_2(1-t)dt, \end{aligned}$$

where $\mathtt{M}(\delta_1, \delta_2)$ and $\mathtt{N}(\delta_1, \delta_2)$ are defined in Theorem 1.1.

In 2022, Sahoo et.al [13] introduced the new variant of Pachpatte-type inequalities via CR-preinvex functions in interval analysis as follows:

Theorem 1.11. *[13] Let* $\Pi_1, \Pi_2 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$*, is an interval functions and* $\Pi_1, \Pi_2 \in IR_{([\delta_2, \delta_1])}$*. If* $\Pi_1, \Pi_2 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$ *are CR-preinvex functions, then the following inequalities hold true:*

$$\frac{1}{\eta(\delta_2, \delta_1)} \int_{\delta_1}^{\delta_1+\eta(\delta_2,\delta_1)} \Pi_1(z)\Pi_2(z)dz \leq_{CR} \frac{\mathtt{M}(\delta_1, \delta_2)}{3} + \frac{\mathtt{N}(\delta_1, \delta_2)}{6},$$

where $\mathtt{M}(\delta_1, \delta_2)$ *and* $\mathtt{N}(\delta_1, \delta_2)$ *are defined in Theorem 1.1.*

Theorem 1.12. *[13] Let* $\Pi_1, \Pi_2 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$*, is an interval functions and* $\Pi_1, \Pi_2 \in IR_{([\delta_2, \delta_1])}$*. If* $\Pi_1, \Pi_2 : [\delta_1, \delta_1 + \eta(\delta_2, \delta_1)] \to R$ *are CR-preinvex functions, then the following inequalities hold true:*

$$\begin{aligned} &2\Pi_1\left(\delta_1 + \frac{1}{2}\eta(\delta_2, \delta_1)\right)\Pi_2\left(\delta_1 + \frac{1}{2}\eta(\delta_2, \delta_1)\right) \\ &\leq_{CR} \frac{1}{\eta(\delta_2, \delta_1)} \int_{\delta_1}^{\delta_1+\eta(\delta_2,\delta_1)} \Pi_1(z)\Pi_2(z)dz + \frac{\mathtt{M}(\delta_1, \delta_2)}{6} + \frac{\mathtt{N}(\delta_1, \delta_2)}{3}, \end{aligned}$$

where $\mathtt{M}(\delta_1, \delta_2)$ *and* $\mathtt{N}(\delta_1, \delta_2)$ *are defined in Theorem 1.1.*

In 2022, Sahoo et.al [13] introduced the new variant of Pachpatte-type inequalities via CR-convex functions in interval analysis as follows:

Theorem 1.13. *[13] Let* $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R$*, is an interval functions and* $\Pi_1, \Pi_2 \in IR_{([\delta_2, \delta_1])}$*. If* $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R$ *are CR–convex functions, then the following inequalities hold true:*

$$\frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \Pi_1(z)\Pi_2(z)dz \leq_{CR} \frac{\mathtt{M}(\delta_1, \delta_2)}{3} + \frac{\mathtt{N}(\delta_1, \delta_2)}{6},$$

where $\mathtt{M}(\delta_1, \delta_2)$ *and* $\mathtt{N}(\delta_1, \delta_2)$ *are defined in Theorem 1.1.*

Theorem 1.14. *[13] Let* $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R$*, is an interval functions and* $\Pi_1, \Pi_2 \in IR_{([\delta_2, \delta_1])}$*. If* $\Pi_1, \Pi_2 : [\delta_1, \delta_2] \to R$ *are CR–convex functions, then the following inequalities hold true:*

$$2\Pi_1\left(\frac{\delta_1 + \delta_2}{2}\right)\Pi_2\left(\frac{\delta_1 + \delta_2}{2}\right) \leq_{CR} \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \Pi_1(z)\Pi_2(z)dz + \frac{\mathtt{M}(\delta_1, \delta_2)}{6} + \frac{\mathtt{N}(\delta_1, \delta_2)}{3},$$

where $\mathtt{M}(\delta_1, \delta_2)$ *and* $\mathtt{N}(\delta_1, \delta_2)$ *are defined in Theorem 1.1.*

2. CONCLUSIONS

Convexity theory allows us to create new, innovative numerical model frameworks that may be used to tackle a wide range of problems in the pure and applied sciences. Thus, convex analysis and its associated inequalities are growing in academic attention and appeal due to several advancements, modifications, and applications. Presenting and offering a comprehensive and current assessment of Pachpatte-type inequalities relevant to different classes of convexities was our goal in this review work. The theoretical and practical significance of Pachpatte-type inequalities was taken into consideration when preparing this review. We think the current review will inspire and give scholars studying Pachpatte-type disparities a place to learn about previous research on the subject before coming up with new findings. It would be interesting to conduct more research on this review paper. Numerous such investigations may be inspired by our review paper.

Conflicts of Interest

The authors declare no conflicts of interest.

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