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A Comprehensive Review Note on Pachpatte-Type Inequality

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Abstract

The term convexity is primarily utilized to address several challenges in both pure and applied research. The main focus and objective of this review paper is to present the Pachpatte-type inequalities via different type of convexities.

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1 Introduction

Convex functions have a long and illustrious history. The history of convexity theory can be traced all the way back to the end of the nineteenth century. The word ”convex functions” has been widely used and explored in the well-known and popular book namely ”Inequalities,” published by G. Polya, G.H. Hardy, and J.E. Littlewood [1]. This book is quickly became a standard reference for mathematicians and dedicated solely to the topic of inequality and serves as an excellent introduction to this fascinating field. Convex theory provides us with appropriate guidelines and techniques to focus on a broad range of problems in applied sciences. It has been widely acknowledged in recent years that mathematical inequalities have contributed to the development of various aspects of mathematics as well as other scientific disciplines. This theory have remarkable uses in economics [2], optimization [3], mathematical optimization for modeling [4, 5], finance [6], computer science [7], control systems [8] and estimation and signal processing [9]. This theory provides a solid framework for the start and growth of numerical tools for the study and solution of challenging mathematical issues.

The investigation of inequality has been regarded and viewed as one of the key fields of ap-plied sciences. It is a fast-growing science with a rising number of applications in many scientific domains. Integral inequalities have fruitful importance in integral operator theory, stochastic processes, numerical integration, optimization theory, probability, statistics and information technology.

The combined study of convex analysis and integral inequalities presents a captivating and engrossing field of research within the area of mathematical interpretation. Because of their widespread perspectives and applications, the tactics and literature of convex analysis and integral inequalities have recently become the subject of intensive research in both contemporary and historical times.

Our objective in this paper is to present a comprehensive and up-to-date review of Pachpatte-type inequalities for different kinds of convexities. We believe that the collection of almost all existing in the literature Pachpatte-type inequalities in one file will help new researchers in the field learn about the available work on the topic before developing new results. We present the results without proof but instead, provide a complete reference for the details of each result elaborated in this survey for the convenience of the reader.

Note that the main motivation of this review paper is to clarify the state of knowledge, explain apparent contradictions, identify needed research, and even create a consensus where none previously existed. The purpose of this review paper is to succinctly review recent progress in a particular topic, namely convex analysis in the frame of fractional calculus. Overall, this review paper summarizes the current state of knowledge on the topic of convexity. It creates an understanding of the topic for the reader by discussing the findings presented in recent research papers. Our goal here is a more complete and comprehensive review and, as such, the choice is made to include as many results as possible to illustrate the progress on the matter. Any proofs (which are rather long) are omitted; for this matter, the reader is accordingly referred to the relevant article.

Mathematical inequalities play important roles in the study of mathematics as well as in other areas of mathematics because of their wide applications in mathematics and physics. One of the most significant functions used to study many interesting inequalities is the convex function, which is defined as follows:

Let $\mathbb{I} \subset \mathbb{R}$ be a non-empty interval. The function $\mathtt{F} : \mathbb{I} \to \mathbb{R}$ is called convex if

$$\mathtt{F}(\lambda \mathtt{u}_1 + (1-\lambda)\mathtt{u}_2) \le \lambda \mathtt{F}(\mathtt{u}_1) + (1-\lambda)\mathtt{F}(\mathtt{u}_2),$$

holds for every $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{I}$ and $\lambda \in [0,1]$.

Currently, many researchers have been fascinated by the field of convex functions and, particularly, one of the well-known inequalities for convex functions known as the Pachpatte-type inequality, which is defined as follows: Pachpatte type inequality [10] asserts that:

Theorem 1.1. *[10] If $\mathtt{F}_1$ and $\mathtt{F}_2$ be a non-negative real-valued convex function on* $[\mathtt{u}_1, \mathtt{u}_2]$, *then*

$$\frac{1}{\mathtt{u}_2 - \mathtt{u}_1} \int_{\mathtt{u}_1}^{\mathtt{u}_2} \mathtt{F}_1(\wp)\mathtt{F}_2(\wp) d\wp \le \frac{1}{3}\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) + \frac{1}{6}\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2),$$

and

$$2\mathtt{F}_1\left(\frac{\mathtt{u}_1 + \mathtt{u}_2}{2}\right)\mathtt{F}_2\left(\frac{\mathtt{u}_1 + \mathtt{u}_2}{2}\right) \le \frac{1}{\mathtt{u}_2 - \mathtt{u}_1} \int_{\mathtt{u}_1}^{\mathtt{u}_2} \mathtt{F}_1(\wp)\mathtt{F}_2(\wp) d\wp + \frac{1}{3}\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) + \frac{1}{6}\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2),$$

where $\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{M}(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2), \quad \mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{M}(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_1)$.

In 2007, Kirmaci et.al [11] introduced the new variant of Pachpatte-type inequalities via s-convex functions in classical calculus as follows:

Theorem 1.2. *[11] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R, \mathtt{u}_1, \mathtt{u}_2 \in [0, \infty)$, $\mathtt{u}_1 < \mathtt{u}_2$ *be functions such that* $\mathtt{F}_2, \mathtt{F}_1\mathtt{F}_2 \in L[\mathtt{u}_1, \mathtt{u}_2]$ *. If* $\mathtt{F}_1$ *is convex and non-negative on* $[\mathtt{u}_1, \mathtt{u}_2]$ *and* $\mathtt{F}_2$ *is s-convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *for some fixed* $s \in (0, 1]$, *then*

$$\frac{1}{\mathtt{u}_2 - \mathtt{u}_1} \int_{\mathtt{u}_1}^{\mathtt{u}_2} \mathtt{F}_1(x)\mathtt{F}_2(x) dx \le \frac{1}{s+2}\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) + \frac{1}{(s+1)(s+2)}\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2),$$

where

$$\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2)$$

and

$$\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

Theorem 1.3. *[11] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R, \mathtt{u}_1, \mathtt{u}_2 \in [0, \infty)$, $\mathtt{u}_1 < \mathtt{u}_2$ *be functions such that* $\mathtt{F}_1, \mathtt{F}_2$ *and* $\mathtt{F}_1\mathtt{F}_2 \in L[\mathtt{u}_1, \mathtt{u}_2]$ *. If* $\mathtt{F}_1$ *is* s_1*-convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *and* $\mathtt{F}_2$ *is* s_2*-convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *for some fixed* $s_1, s_1 \in (0, 1]$*, then*

$$\frac{1}{\mathtt{u}_2 - \mathtt{u}_1} \int_{\mathtt{u}_1}^{\mathtt{u}_2} \mathtt{F}_1(x)\mathtt{F}_2(x)dx \leq \frac{1}{s_1 + s_2 + 1}\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) + \beta(s_2 + 1, s_1 + 1)\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2),$$

where

$$\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2)$$

and

$$\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

Theorem 1.4. *[11] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R, \mathtt{u}_1, \mathtt{u}_2 \in [0, \infty)$, $\mathtt{u}_1 < \mathtt{u}_2$ *be functions such that* $\mathtt{F}_1, \mathtt{F}_2$ *and* $\mathtt{F}_1\mathtt{F}_2 \in L[\mathtt{u}_1, \mathtt{u}_2]$ *. If* $\mathtt{F}_1$ *is convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *and* $\mathtt{F}_2$ *is* s*-convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *for some fixed* $s \in (0, 1]$*, then*

$$\begin{aligned}
&2^s\mathtt{F}_1\left(\frac{\mathtt{u}_1 + \mathtt{u}_2}{2}\right)\mathtt{F}_2\left(\frac{\mathtt{u}_1 + \mathtt{u}_2}{2}\right) \\
&\leq \frac{1}{\mathtt{u}_2 - \mathtt{u}_1} \int_{\mathtt{u}_1}^{\mathtt{u}_2} \mathtt{F}_1(x)\mathtt{F}_2(x)dx + \frac{1}{(s+1)(s+2)}\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) + \frac{1}{s+2}\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2),
\end{aligned}$$

where

$$\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

In 2014, Chen et.al [12] introduced a new variant of Pachpatte-type inequalities via harmonically s-convex functions in classical calculus as follows:

Theorem 1.5. *[12] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R, \mathtt{u}_1, \mathtt{u}_2 \in [0, \infty)$, $\mathtt{u}_1 < \mathtt{u}_2$ *be functions such that* $\mathtt{F}_1, \mathtt{F}_2$ *and* $\mathtt{F}_1\mathtt{F}_2 \in L[\mathtt{u}_1, \mathtt{u}_2]$ *. If* $\mathtt{F}_1$ *is harmonically* s_1*-convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *and* $\mathtt{F}_2$ *is harmonically* s_2*-convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *for some fixed* $s_1, s_1 \in (0, 1]$*, then*

$$\frac{\mathtt{u}_1\mathtt{u}_2}{\mathtt{u}_2 - \mathtt{u}_1} \int_{\mathtt{u}_1}^{\mathtt{u}_2} \frac{\mathtt{F}_1(x)\mathtt{F}_2(x)}{x^2}dx \leq \frac{1}{1 + s_1 + s_2}\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) + \frac{\Gamma(1 + s_1)\Gamma(1 + s_2)}{\Gamma(s_1 + s_2 + 2)}\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2),$$

where

$$\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2)$$

and

$$\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

Theorem 1.6. *[12] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R, \mathtt{u}_1, \mathtt{u}_2 \in [0, \infty)$, $\mathtt{u}_1 < \mathtt{u}_2$ *be functions such that* $\mathtt{F}_1, \mathtt{F}_2$ *and* $\mathtt{F}_1\mathtt{F}_2 \in L[\mathtt{u}_1, \mathtt{u}_2]$ *. If* $\mathtt{F}_1$ *is harmonically* s_1*-convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *and* $\mathtt{F}_2$ *is harmonically* s_2*-convex on* $[\mathtt{u}_1, \mathtt{u}_2]$ *for some fixed* $s_1, s_1 \in (0, 1]$*, then*

$$\begin{aligned}
&2^{s_1+s_2-1}\mathtt{F}_1\left(\frac{2\mathtt{u}_1\mathtt{u}_2}{\mathtt{u}_1 + \mathtt{u}_2}\right)\mathtt{F}_2\left(\frac{2\mathtt{u}_1\mathtt{u}_2}{\mathtt{u}_1 + \mathtt{u}_2}\right) \\
&\leq \frac{\mathtt{u}_1\mathtt{u}_2}{\mathtt{u}_2 - \mathtt{u}_1} \int_{\mathtt{u}_1}^{\mathtt{u}_2} \frac{\mathtt{F}_1(x)\mathtt{F}_2(x)}{x^2}dx + \mathtt{M}(\mathtt{u}_1, \mathtt{u}_2)\frac{\Gamma(1 + s_1)\Gamma(1 + s_2)}{\Gamma(s_1 + s_2 + 2)} + \mathtt{N}(\mathtt{u}_1, \mathtt{u}_2)\frac{1}{1 + s_1 + s_2},
\end{aligned}$$

where

$$\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2)$$

and

$$\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

In 2022, Sahoo et.al [13] introduced the new variant of Pachpatte-type inequalities via CR-h-preinvex functions in interval analysis as follows:

Theorem 1.7. *[13] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_1 + \eta(\mathtt{u}_2, \mathtt{u}_1)] \to R$, *is an interval functions and* $\mathtt{F}_1, \mathtt{F}_2 \in IR_{([\mathtt{u}_2, \mathtt{u}_1])}$. *If* $\mathtt{F}_1 : [\mathtt{u}_1, \mathtt{u}_1 + \eta(\mathtt{u}_2, \mathtt{u}_1)] \to R$ *is a CR-*h_1*-preinvex function and* $\mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_1 + \eta(\mathtt{u}_2, \mathtt{u}_1)] \to R$ *is a CR-*h_2*-preinvex function, then the following inequalities hold true:*

$$\begin{aligned}&\frac{1}{\eta(\mathtt{u}_2, \mathtt{u}_1)} \int_{\mathtt{u}_1}^{\mathtt{u}_1 + \eta(\mathtt{u}_2, \mathtt{u}_1)} \mathtt{F}_1(z)\mathtt{F}_2(z)dz \\ &\leq_{CR} \mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) \int_0^1 h_1(1-t)h_2(1-t)dt + \mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) \int_0^1 h_1(1-t)h_2(t)dt,\end{aligned}$$

where

$$\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

Theorem 1.8. *[13] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_1 + \eta(\mathtt{u}_2, \mathtt{u}_1)] \to R$, *is an interval functions and* $\mathtt{F}_1, \mathtt{F}_2 \in IR_{([\mathtt{u}_2, \mathtt{u}_1])}$. *If* $\mathtt{F}_1 : [\mathtt{u}_1, \mathtt{u}_1 + \eta(\mathtt{u}_2, \mathtt{u}_1)] \to R$ *is a CR-*h_1*-preinvex function and* $\mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_1 + \eta(\mathtt{u}_2, \mathtt{u}_1)] \to R$ *is a CR-*h_2*-preinvex function, then the following inequalities hold true:*

$$\begin{aligned}&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\mathtt{F}_1\left(\mathtt{u}_1 + \frac{1}{2}\eta(\mathtt{u}_2, \mathtt{u}_1)\right)\mathtt{F}_2\left(\mathtt{u}_1 + \frac{1}{2}\eta(\mathtt{u}_2, \mathtt{u}_1)\right) \\ &\leq_{CR} \frac{1}{\eta(\mathtt{u}_2, \mathtt{u}_1)} \int_{\mathtt{u}_1}^{\mathtt{u}_1 + \eta(\mathtt{u}_2, \mathtt{u}_1)} \mathtt{F}_1(z)\mathtt{F}_2(z)dz \\ &+ \mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) \int_0^1 h_1(1-t)h_2(t)dt + \mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) \int_0^1 h_1(1-t)h_2(1-t)dt,\end{aligned}$$

where

$$\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

In 2022, Sahoo et.al [13] introduced the new variant of Pachpatte-type inequalities via CR-h-convex functions in interval analysis as follows:

Theorem 1.9. *[13] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R$, *is an interval functions and* $\mathtt{F}_1, \mathtt{F}_2 \in IR_{([\mathtt{u}_2, \mathtt{u}_1])}$. *If* $\mathtt{F}_1 : [\mathtt{u}_1, \mathtt{u}_2] \to R$ *is a CR-*h_1*-convex function and* $\mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R$ *is a CR-*h_2*-convex function, then the following inequalities hold true:*

$$\begin{aligned}&\frac{1}{\mathtt{u}_2 - \mathtt{u}_1} \int_{\mathtt{u}_1}^{\mathtt{u}_2} \mathtt{F}_1(z)\mathtt{F}_2(z)dz \\ &\leq_{CR} \mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) \int_0^1 h_1(1-t)h_2(1-t)dt + \mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) \int_0^1 h_1(1-t)h_2(t)dt,\end{aligned}$$

where

$$\mathtt{M}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1, \mathtt{u}_2) = \mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2) + \mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

Theorem 1.10. *[13] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R$*, is an interval functions and* $\mathtt{F}_1, \mathtt{F}_2 \in IR_{([\mathtt{u}_2, \mathtt{u}_1])}$*. If* $\mathtt{F}_1 : [\mathtt{u}_1, \mathtt{u}_2] \to R$ *is a CR-*h_1*-convex function and* $\mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R$ *is a CR-*h_2*-convex function, then the following inequalities hold true:*

$$\begin{aligned}&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\mathtt{F}_1\left(\frac{\mathtt{u}_1+\mathtt{u}_2}{2}\right)\mathtt{F}_2\left(\frac{\mathtt{u}_1+\mathtt{u}_2}{2}\right)\\ &\leq_{CR} \frac{1}{\mathtt{u}_2-\mathtt{u}_1}\int_{\mathtt{u}_1}^{\mathtt{u}_2}\mathtt{F}_1(z)\mathtt{F}_2(z)dz\\ &+\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)\int_0^1 h_1(1-t)h_2(t)dt+\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)\int_0^1 h_1(1-t)h_2(1-t)dt,\end{aligned}$$

where

$$\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

In 2022, Sahoo et.al [13] introduced the new variant of Pachpatte-type inequalities via CR-preinvex functions in interval analysis as follows:

Theorem 1.11. *[13] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_1+\eta(\mathtt{u}_2,\mathtt{u}_1)] \to R$*, is an interval functions and* $\mathtt{F}_1, \mathtt{F}_2 \in IR_{([\mathtt{u}_2, \mathtt{u}_1])}$*. If* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_1+\eta(\mathtt{u}_2,\mathtt{u}_1)] \to R$ *are CR-preinvex functions, then the following inequalities hold true:*

$$\frac{1}{\eta(\mathtt{u}_2,\mathtt{u}_1)}\int_{\mathtt{u}_1}^{\mathtt{u}_1+\eta(\mathtt{u}_2,\mathtt{u}_1)}\mathtt{F}_1(z)\mathtt{F}_2(z)dz \leq_{CR} \frac{\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)}{3}+\frac{\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)}{6},$$

where

$$\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

Theorem 1.12. *[13] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_1+\eta(\mathtt{u}_2,\mathtt{u}_1)] \to R$*, is an interval functions and* $\mathtt{F}_1, \mathtt{F}_2 \in IR_{([\mathtt{u}_2, \mathtt{u}_1])}$*. If* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_1+\eta(\mathtt{u}_2,\mathtt{u}_1)] \to R$ *are CR-preinvex functions, then the following inequalities hold true:*

$$\begin{aligned}&2\mathtt{F}_1\left(\mathtt{u}_1+\frac{1}{2}\eta(\mathtt{u}_2,\mathtt{u}_1)\right)\mathtt{F}_2\left(\mathtt{u}_1+\frac{1}{2}\eta(\mathtt{u}_2,\mathtt{u}_1)\right)\\ &\leq_{CR} \frac{1}{\eta(\mathtt{u}_2,\mathtt{u}_1)}\int_{\mathtt{u}_1}^{\mathtt{u}_1+\eta(\mathtt{u}_2,\mathtt{u}_1)}\mathtt{F}_1(z)\mathtt{F}_2(z)dz+\frac{\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)}{6}+\frac{\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)}{3},\end{aligned}$$

where

$$\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

In 2022, Sahoo et.al [13] introduced the new variant of Pachpatte-type inequalities via CR-convex functions in interval analysis as follows:

Theorem 1.13. *[13] Let* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R$*, is an interval functions and* $\mathtt{F}_1, \mathtt{F}_2 \in IR_{([\mathtt{u}_2, \mathtt{u}_1])}$*. If* $\mathtt{F}_1, \mathtt{F}_2 : [\mathtt{u}_1, \mathtt{u}_2] \to R$ *are CR–convex functions, then the following inequalities hold true:*

$$\frac{1}{\mathtt{u}_2-\mathtt{u}_1}\int_{\mathtt{u}_1}^{\mathtt{u}_2}\mathtt{F}_1(z)\mathtt{F}_2(z)dz \leq_{CR} \frac{\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)}{3}+\frac{\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)}{6},$$

where

$$\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

Theorem 1.14. *[13] Let $\mathtt{F}_1,\mathtt{F}_2:[\mathtt{u}_1,\mathtt{u}_2]\to R$, is an interval functions and $\mathtt{F}_1,\mathtt{F}_2\in IR_{([\mathtt{u}_2,\mathtt{u}_1])}$. If $\mathtt{F}_1,\mathtt{F}_2:[\mathtt{u}_1,\mathtt{u}_2]\to R$ are CR–convex functions, then the following inequalities hold true:*

$$2\mathtt{F}_1\left(\frac{\mathtt{u}_1+\mathtt{u}_2}{2}\right)\mathtt{F}_2\left(\frac{\mathtt{u}_1+\mathtt{u}_2}{2}\right)\leq_{CR}\frac{1}{\mathtt{u}_2-\mathtt{u}_1}\int_{\mathtt{u}_1}^{\mathtt{u}_2}\mathtt{F}_1(z)\mathtt{F}_2(z)dz+\frac{\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)}{6}+\frac{\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)}{3},$$

where

$$\mathtt{M}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_1)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2),$$

and

$$\mathtt{N}(\mathtt{u}_1,\mathtt{u}_2)=\mathtt{F}_1(\mathtt{u}_1)\mathtt{F}_2(\mathtt{u}_2)+\mathtt{F}_1(\mathtt{u}_2)\mathtt{F}_2(\mathtt{u}_2).$$

2 Conclusions

Convexity theory allows us to create new, innovative numerical model frameworks that may be used to tackle a wide range of problems in the pure and applied sciences. Thus, convex analysis and its associated inequalities are growing in academic attention and appeal due to several advancements, modifications, and applications. Our objective in this review paper was to present and provide a comprehensive and up-to-date review on Pachpatte-type inequalities pertaining to various classes of convexities. This review was prepared to keep in mind the theoretical and practical importance of the Pachpatte-type inequalities. We believe that the present review will motivate and provide a platform for the researchers working on Pachpatte-type inequalities to learn about the available work on the topic before developing new results. Future research regarding this review paper is fascinating. Our review paper might inspire a good number of additional studies.

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