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Hermite-Hadamard Type Inequality and its Applications via Modified Riemann Liouville Fractional Integral Operator

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Abstract

The main aim of this manuscript is to investigate a new form of Hermite-Hadamard inequalities via ψ-Riemann-Liouville fractional integrals for preinvex functions. By em-ploying this approach, we construct a new fractional integral identity that correlates with preinvex functions. In addition, based on this newly derived fractional identity, some new estimation of fractional Hermite-Hadamard type inequality involving m-preinvex via ψ-Riemann-Liouville fractional sense is investigated. Further, we pointed out some appli-cations for special means.

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1 Introduction

Convex functions have a long and illustrious history. The history of convexity theory can be traced all the way back to the end of the nineteenth century. Convex theory provides us with appropriate guidelines and techniques to focus on a broad range of problems in applied sciences. It has been widely acknowledged in recent years that mathematical inequalities have contributed to the development of various aspects of mathematics as well as other scientific disciplines.

The term invex function first time examined by Hanson [1]. Mond and Weir [2] explored the notion of preinvexity. The analysis of the preinvex and invex theory utilizing the bifunction by Mond and Ben-Israel [3] can be viewed as a significant contribution to the optimization field.

The aim and novelty of this work are to introduce a new variant of H-H type integral inequality via preinvexity in the frame of $\Psi - \mathrm{RLF\ IO}$. Further, we are to construct some refinements of H-H type integral inequality via Ψ-RLFIO..

2 Preliminaries

In this section, we recall some basic definitions and results required for this manuscript.

Definition 2.1 ([4])**.** *$\mathbb{X} \subset \mathbb{R}^n$ is invex w.r.t $\Omega(.,.)$, if*

$$\flat_1 + \wp\Omega(\flat_2, \flat_1) \in \mathbb{X},$$

$\forall\ \flat_1, \flat_2 \in \mathbb{X}$ and $\wp \in [0,1]$.

Definition 2.2 ([5])**.** *Assume that $\Omega : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}^n$ and $\mathbb{X} \subseteq \mathbb{R}^n$. Then $\mathbb{X}$ is m–invex w.r.t. Ω, if*

$$m\flat_2 + \wp\Omega(\flat_1, \flat_2, m) \in \mathbb{X}$$

holds $\forall\ \flat_1, \flat_2 \in \mathbb{X}$, $m \in (0,1]$ and $\wp \in [0,1]$.

Example 2.1 ([5])**.** *Assume that $m = \frac{1}{4}$, $\mathbb{X} = [\frac{-\pi}{2}, 0) \bigcup (0, \frac{1}{2}]$ and*

$$\Omega(\flat_2, \flat_1, m) = \begin{cases} m\cos(\flat_2 - \flat_1) & if\ \flat_1 \in (0, \frac{\pi}{2}], \flat_2 \in (0, \frac{\pi}{2}]; \\ -m\cos(\flat_2 - \mu_1) & if\ \flat_1 \in [\frac{-\pi}{2}, 0), \flat_2 \in [\frac{-\pi}{2}, 0); \\ m\cos(\flat_1) & if\ \flat_1 \in (0, \frac{\pi}{2}], \flat_2 \in [\frac{-\pi}{2}, 0); \\ -m\cos(\flat_1) & if\ \flat_1 \in [-\frac{\pi}{2}, 0), \flat_2 \in (0, \frac{\pi}{2}]. \end{cases}$$

Then, $\mathbb{X}$ is an m–invex set but not convex $\forall\ \wp \in [0,1]$.

Definition 2.3 ([2])**.** *Assume that $\Omega : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$ and $\mathbb{X} \subseteq \mathbb{R}^n$. Then $\mathcal{F} : \mathbb{X} \to \mathbb{R}$ is preinvex w.r.t Ω if*

$$\mathcal{F}\left(\flat_2 + \wp\Omega\left(\flat_1, \flat_2\right)\right) \ \leq\ \wp\ \mathcal{F}\left(\flat_1\right)\ +\ \left(1 - \wp\right)\ \mathcal{F}\left(\flat_2\right)\ , \qquad \forall \flat_1,\ \flat_2 \in \mathbb{X}\ , \quad \wp \in [0,1]\,.$$

Definition 2.4. *[6] Assume that $\Omega : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}^n$ and $\mathbb{X} \subseteq \mathbb{R}^n$. Then $\mathcal{F} : \mathbb{X} \to \mathbb{R}$ is generalized m-preinvex w.r.t. Ω if*

$$\mathcal{F}(m\flat_2 + \wp\Omega(\flat_1, \flat_2, m)) \leq \wp\mathcal{F}\left(\flat_1\right) + m\left(1 - \wp\right)\mathcal{F}\left(\flat_2\right), \tag{2.1}$$

holds for every $\flat_1, \flat_2 \in \mathbb{X}$, $m \in (0,1]$ and $\wp \in [0,1]$.

Condition C: Assume that $\mathbb{X} \subset \mathbb{R}$ is an open invex subset w.r.t. $\Omega : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. We say that the function Ω satisfied the condition C, if for any $\flat_1, \flat_2 \in \mathbb{X}$ and $\wp \in [0,1]$,

$$\begin{aligned} \Omega\left(\flat_1, \flat_1 + \wp\ \Omega\left(\flat_2, \flat_1\right)\right) &= \ -\wp\ \Omega\left(\flat_2, \flat_1\right) \\ \Omega\left(\flat_2, \flat_1 + \wp\ \Omega\left(\flat_2, \flat_1\right)\right) &= \ \left(1 - \wp\right)\ \Omega\left(\flat_2, \flat_1\right). \end{aligned}$$

For any $\flat_1, \flat_2 \in \mathbb{X}$, $\wp_1, \wp_2 \in [0,1]$, then according to the above equations, we have

$$\Omega\left(\flat_1 + \wp_2\ \Omega\left(\flat_2, \flat_1\right),\ \flat_1 + \wp_1\ \Omega\left(\flat_2, \flat_1\right)\right) = \ \left(\wp_2 - \wp_1\right)\Omega\left(\flat_2, \flat_1\right).$$

This Condition is very important in the optimization and creation of the theory of inequalities (see [7, 8]).

The following extended Condition C in the frame of m-preinvexity was also discussed by Du et.al in [9].

Extended Condition C: Assume that $\mathbb{X} \subset \mathbb{R}$ an open invex subset w.r.t. $\Omega : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}$. We say that the function Ω satisfied the Extended Condition C, if for any $\flat_1, \flat_2 \in \mathbb{X}$, $\wp \in [0,1]$, we have

$$\begin{aligned} \Omega\left(\flat_2, m\flat_2 + \wp\ \Omega\left(\flat_1, \flat_2, m\right), m\right) &= -\wp\ \Omega\left(\flat_1, \flat_2, m\right) \\ \Omega\left(\flat_1, m\flat_2 + \wp\ \Omega\left(\flat_1, \flat_2, m\right), m\right) &= \left(1 - \wp\right)\ \Omega\left(\flat_1, \flat_2, m\right) \\ \Omega\left(\flat_1, \flat_2, m\right) &= -\Omega\left(\flat_2, \flat_1, m\right). \end{aligned}$$

Theorem 2.1. *[10] Assume that $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Assume thats $\Psi_1, \Psi_2 : [x_1, x_2] \to \mathbb{R}$ are such that $|\Psi_1|^p$ and $|\Psi_2|^q$ are integrable on $[x_1, x_2]$. Then*

$$\int_0^1 |\mathcal{F}_1(x)\mathcal{F}_2(x)|dx \leq \Big(\int_0^1 |\mathcal{F}_1(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_0^1 |\mathcal{F}_2(x)|^q dx\Big)^{\frac{1}{q}}.$$

Many mathematicians with the development of fractional calculus have defined many fractional derivative and integral operators to find solutions to real-world problems. Some of them are as follows.

Here, we recall the Ψ as follows:

Definition 2.5. *[11] Let $(\flat_1, \flat_2)(-\infty \leq \flat_1 < \flat_2 \leq \infty)$ be an interval of the real line R and $\alpha > 0$. Also let $\psi(w)$ be an increasing and positive monotone function on $(\flat_1, \flat_2]$, having a continuous derivative $\psi^{'}(w)$ on $(\flat_1, \flat_2])$. The left and right-sided ψ-R-L fractional integrals of a function $\mathcal{F}$ with respect to an other function ψ on $(\flat_1, \flat_2)$. are defined by*

$$I^{\alpha:\psi}_{\flat_1^+}\mathcal{F}(w) = \frac{1}{\Gamma(\alpha)}\int_{\flat_1}^{w} \psi'(\mu)(\psi(w)-\psi(\mu))^{\alpha-1}\mathcal{F}(\mu)d\mu,$$
$$I^{\alpha:\psi}_{\flat_1^-}\mathcal{F}(w) = \frac{1}{\Gamma(\alpha)}\int_{w}^{\flat_2} \psi'(\mu)(\psi(\mu)-\psi(w))^{\alpha-1}\mathcal{F}(\mu)d\mu,$$

respectively.

3 Hermite–Hadamard Inequality via Ψ-Riemann-Liouville Fractional Integral Operator

The main goal of this portion is to provide a new sort of the H–H-type inequality for a m-preinvex function via Ψ-RLFIO.

Theorem 3.1. *Let $I \subseteq \mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega : I \times I \to \mathbb{R}$ and $\flat_1, \flat_2 \in I$ with $m\flat_1 < m\flat_1 + \Omega(\flat_2, \flat_1, m)$. If $\mathcal{F} : [m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m)] \to \mathbb{R}$ is a m-preinvex function and $\mathcal{F} \in L[m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m)]$ and Ω satisfies extended condition C. Also suppose $\Psi(w)$ is an increasing and positive function on $(m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m)$, having a continous drivative $\Psi(w)^{'}$ on $(m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m)$ and $\alpha \in (0, 1)$, then*

$$\begin{aligned}
&\mathcal{F}(m\flat_1 + \frac{1}{2}\Omega(\flat_2, \flat_1, m)) \\
&\leq \frac{\Gamma(\alpha+1)}{2\Omega^{\alpha}(\flat_2, \flat_1, m)}[I^{\alpha:\psi}_{\psi^{-}(m\flat_1)^{+}}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1 + \Omega(\flat_2, \flat_1, m))] \\
&+ [I^{\alpha:\psi}_{\psi^{-}(m\flat_1+\Omega(\flat_2,\flat_1,m))^{-}}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1)] \\
&\leq \frac{\mathcal{F}(m\flat_1) + \mathcal{F}(m\flat_1 + \Omega(\flat_2, \flat_1, m)}{2} \leq \frac{\mathcal{F}(m\flat_1) + \mathcal{F}(\flat_2)}{2}.
\end{aligned} \tag{3.1}$$

Proof. Since $\mathcal{F}$ is m-preinvex function on $[m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m)]$, we can write

$$\mathcal{F}(mw + \frac{1}{2}\Omega(z, w, m)) \leq \frac{\mathcal{F}(mw) + \mathcal{F}(z)}{2}.$$

Using $w = m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m)$ and $z = m\flat_1 + \delta\Omega(\flat_2, \flat_1, m)$ in 3.1 we have

$$\begin{aligned}
&\mathcal{F}(m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m) + \frac{1}{2}\Omega(m\flat_1 + \delta\Omega(\flat_2, \flat_1, m), m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m))) \\
&\leq \frac{\mathcal{F}(m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m)) + \mathcal{F}(m\flat_1 + \delta\Omega(\flat_2, \flat_1, m)}{2}.
\end{aligned} \tag{3.2}$$

Applying extended Condition C in 3.2, we have

$$\mathcal{F}(m\flat_1 + \frac{1}{2}\Omega(\flat_2, \flat_1, m)) \\ \leq \frac{\mathcal{F}(m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m)) + \mathcal{F}(m\flat_1 + \delta\Omega(\flat_2, \flat_1, m))}{2}. \tag{3.3}$$

Multiplying both sides of the above inequality (3.3) by $\delta^{\alpha-1}$ then integrating the resulting inequality with respect to δ over $[0, 1]$, we obtain

$$\frac{2}{\alpha}\mathcal{F}(m\flat_1 + \frac{1}{2}\Omega(\flat_2, \flat_1, m)) \\ \leq \int_0^1 \delta^{\alpha-1}\mathcal{F}(m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m))d\delta + \int_0^1 \delta^{\alpha-1}\mathcal{F}(m\flat_1 + \delta\Omega(\flat_2, \flat_1, m))d\delta.$$

Next

$$\frac{\Gamma(\alpha+1)}{2\Omega^\alpha(\flat_2, \flat_1, m)}[I^{\alpha:\psi}_{\psi^-(m\flat_1)^+}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1 + \Omega(\flat_2, \flat_1, m))] + [I^{\alpha:\psi}_{\psi^-(m\flat_1+\Omega(\flat_2,\flat_1,m))^-}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1)] \\ = \frac{\alpha}{2\Omega^\alpha(\flat_2, \flat_1, m)}[\int_{\psi^{-1}(m\flat_1)}^{\psi^{-1}(m\flat_1+\Omega^\alpha(\flat_2,\flat_1,m)}(m\flat_1 + \Omega(\flat_2, \flat_1, m) - \psi(\mu))^{\alpha-1}(\mathcal{F}o\psi)(\mu)\psi'(\mu)d\mu \\ + \int_{\psi^{-1}(m\flat_1)}^{\psi^{-1}(m\flat_1+\Omega^\alpha(\flat_2,\flat_1,m)}(\psi(\mu) - m\flat_1)^{\alpha-1}(\mathcal{F}o\psi)(\mu)\psi'(\mu)d\mu] \\ = \frac{\alpha}{2}\int_0^1 \delta^{\alpha-1}\mathcal{F}(m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m))d\delta + \int_0^1 \delta^{\alpha-1}\mathcal{F}(m\flat_1 + \delta\Omega(\flat_2, \flat_1, m))d\delta. \tag{3.4}$$

From the inequalities 3.3 and 3.3, we get

$$\mathcal{F}(m\flat_1 + \frac{1}{2}\Omega(\flat_2, \flat_1, m)) \\ \leq \frac{\Gamma(\alpha+1)}{2\Omega^\alpha(\flat_2, \flat_1, m)}[I^{\alpha:\psi}_{\psi^-(m\flat_1)^+}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1 + \Omega(\flat_2, \flat_1, m))] \\ + [I^{\alpha:\psi}_{\psi^-(m\flat_1+\Omega(\flat_2,\flat_1,m))^-}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1)]$$

and the first inequality is proved.

For the proof of the second inequality, we have

$$\mathcal{F}(m\flat_1 + \delta\Omega(\flat_2, \flat_1, m)) = \mathcal{F}(m\flat_1 + \Omega(\flat_2, \flat_1, m) + (1-\delta)\Omega(m\flat_1, m\flat_1 + \delta\Omega(\flat_2, \flat_1, m))) \\ \leq \mathcal{F}(m\flat_1 + \Omega(\flat_2, \flat_1, m) + (1-\delta)\mathcal{F}(m\flat_1). \tag{3.5}$$

and

$$\mathcal{F}(m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m)) = \mathcal{F}(m\flat_1 + \Omega(\flat_2, \flat_1, m) + \delta\Omega(m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m))) \\ \leq (1-\delta)\mathcal{F}(m\flat_1 + \Omega(\flat_2, \flat_1, m)) + \delta\mathcal{F}(m\flat_1). \tag{3.6}$$

From the inequalities 3.5 and 3.6, we get

$$\mathcal{F}(m\flat_1 + \Omega(\flat_2, \flat_1, m)) + \mathcal{F}(m\flat_1 + (1-\delta)\Omega(\flat_2, \flat_1, m)) + \delta\mathcal{F}(m\flat_1)) \\ \leq \mathcal{F}(m\flat_1) + \mathcal{F}(m\flat_1 + \Omega(\flat_2, \flat_1, m)). \tag{3.7}$$

Then, multiplying both sides of the above inequality (3.7) by $\delta^{\alpha-1}$ and integrating the resulting inequality with respect to δ over $[0,1]$, we obtain

$$\int_0^1 \delta^{\alpha-1}\mathcal{F}(m\flat_1+\delta\Omega(\flat_2,\flat_1,m))d\delta+\int_0^1 \delta^{\alpha-1}\mathcal{F}(m\flat_1+(1-\delta)\Omega(\flat_2,\flat_1,m))d\delta \leq \frac{\mathcal{F}(m\flat_1)+\mathcal{F}(m\flat_1+\Omega(\flat_2,\flat_1,m)}{\alpha}. \tag{3.8}$$

From the inequalities 3.4 and 3.8, we get

$$\frac{\Gamma(\alpha+1)}{2\Omega^{\alpha}(\flat_2,\flat_1,m)}[I^{\alpha:\psi}_{\psi^-(m\flat_1)^+}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1+\Omega(\flat_2,\flat_1,m)) + I^{\alpha:\psi}_{\psi^-(m\flat_1+\Omega(\flat_2,\flat_1,m))^-}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1)] \leq \frac{\mathcal{F}(m\flat_1)+\mathcal{F}(m\flat_1+\Omega(\flat_2,\flat_1,m))}{\alpha} \leq \frac{\mathcal{F}(m\flat_1)+\mathcal{F}(\flat_2)}{2}.$$

So, the proof of this theorem is completed. □

4 Generalization of H–H-Type Inequality via Ψ-Riemann-Liouville Fractional Integral Operator

Lemma 4.1. *Let $I\subseteq\mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega: I\times I\to\mathbb{R}$ and $\flat_1,\flat_2\in I$ with $m\flat_1<m\flat_1+\Omega(\flat_2,\flat_1,m)$. Suppose that $\Psi: I\to\mathbb{R}$ be a differentiable function. If Ψ' is m-preinvx and $\Psi'\in L[m\flat_1,m\flat_1+\Omega(\flat_2,\flat_1,m)]$. Also suppose $\Psi(w)$ is an increasing and positive function on $(m\flat_1,m\flat_1+\Omega(\flat_2,\flat_1,m)$, having a continous drivative $\Psi(w)'$ on $(m\flat_1,m\flat_1+\Omega(\flat_2,\flat_1,m)$ and $\alpha\in(0,1)$, then*

$$\frac{\mathcal{F}(m\flat_1)+\mathcal{F}(m\flat_1+\Omega(\flat_2,\flat_1,m))}{\alpha}-\frac{\Gamma(\alpha+1)}{2\Omega^{\alpha}(\flat_2,\flat_1,m)}[I^{\alpha:\psi}_{\psi^-(m\flat_1)^+}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1+\Omega(\flat_2,\flat_1,m)) + I^{\alpha:\psi}_{\psi^-(m\flat_1+\Omega(\flat_2,\flat_1,m))^-}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1)]$$
$$=\frac{1}{2\Omega^{\alpha}(\flat_2,\flat_1,m)}\int_{\psi^{-1}(m\flat_1)}^{\psi^{-1}(m\flat_1+\Omega^{\alpha}(\flat_2,\flat_1,m)}[(\psi(\mu)-m\flat_1)^{\alpha}-(m\flat_1+\Omega(\flat_2,\flat_1,m)-\psi(\mu))^{\alpha}](\mathcal{F}'o\psi)(\mu)\psi'(\mu)d\mu$$
$$=\frac{\Omega(\flat_2,\flat_1,m)}{2}\int_0^1((1-\delta)^{\alpha}-\delta^{\alpha})\mathcal{F}'(m\flat_1+(1-\delta)\Omega(\flat_2,\flat_1,m))d\delta.$$

where $\alpha\in(0,1],\varphi\in[0,1]$.

Theorem 4.1. *Let $I\subseteq\mathbb{R}$ be an open and non-empty m-invex subset w.r.t. $\Omega: I\times I\to\mathbb{R}$ and $\flat_1,\flat_2\in I$ with $m\flat_1<m\flat_1+\Omega(\flat_2,\flat_1,m)$. Suppose that $\Psi: I\to\mathbb{R}$ be a differentiable function. If Ψ' is m-preinvx and $\Psi'\in L[m\flat_1,m\flat_1+\Omega(\flat_2,\flat_1,m)]$. Also suppose $\Psi(w)$ is an increasing and positive function on $(m\flat_1,m\flat_1+\Omega(\flat_2,\flat_1,m)$, having a continous drivative $\Psi(w)'$ on $(m\flat_1,m\flat_1+\Omega(\flat_2,\flat_1,m)$ and $\alpha\in(0,1)$, then*

$$\left|\frac{\mathcal{F}(m\flat_1)+\mathcal{F}(m\flat_1+\Omega(\flat_2,\flat_1,m))}{2}-\frac{\Gamma(\alpha+1)}{2\Omega^{\alpha}(\flat_2,\flat_1,m)}[I^{\alpha:\psi}_{\psi^{-1}(m\flat_1)^-}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1+\Omega(\flat_2,\flat_1,m)) + [|I^{\alpha:\psi}_{\psi^{-1}(m\flat_1+\Omega(\flat_2,\flat_1,m))^-}(\mathcal{F}o\psi)(m\flat_1)]\right| \leq \frac{\Omega(\flat_2,\flat_1,m)}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{|\mathcal{F}'(m\flat_1)|^q+|\mathcal{F}'(\flat_2)|^q}{2}\right)^{\frac{1}{q}},$$

where $p^{-1}+q^{-1}=1, q>1, \alpha\in(0,1]$.

Proof. By using Lemma 4.1, we get

$$\left|\frac{\mathcal{F}(m\flat_1)+\mathcal{F}(m\flat_1+\Omega(\flat_2,\flat_1,m))}{2}-\frac{\Gamma(\alpha+1)}{2\Omega^{\alpha}(\flat_2,\flat_1,m)}[I^{\alpha:\psi}_{\psi^{-1}(m\flat_1)^-}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1+\Omega(\flat_2,\flat_1,m))\right.$$
$$\left.+[|I^{\alpha:\psi}_{\psi^{-1}(m\flat_1+\Omega(\flat_2,\flat_1,m))^-}(\mathcal{F}o\psi)(m\flat_1)]\right|$$
$$\leq\frac{\Omega(\flat_2,\flat_1,m)}{2}\int_0^1|\delta^{\alpha}-(1-\delta)^{\alpha}||\mathcal{F}'(m\flat_1+\delta\Omega(\flat_2,\flat_1,m))|d\delta.$$

By applying Hölder inequality, we get

$$\left|\frac{\mathcal{F}(m\flat_1)+\mathcal{F}(m\flat_1+\Omega(\flat_2,\flat_1,m))}{2}-\frac{\Gamma(\alpha+1)}{2\Omega^{\alpha}(\flat_2,\flat_1,m)}[I^{\alpha:\psi}_{\psi^{-1}(m\flat_1)^-}(\mathcal{F}o\psi)\psi^{-1}(m\flat_1+\Omega(\flat_2,\flat_1,m))\right.$$
$$\left.+[|I^{\alpha:\psi}_{\psi^{-1}(m\flat_1+\Omega(\flat_2,\flat_1,m))^-}(\mathcal{F}o\psi)(m\flat_1)]\right|$$
$$\leq\frac{\Omega(\flat_2,\flat_1,m)}{2}\left(\int_0^1|\delta^{\alpha}-(1-\delta)^{\alpha}|^p\,d\delta\right)^{\frac{1}{p}}\left(\int_0^1|\mathcal{F}'(m\flat_1+\delta\Omega(\flat_2,\flat_1,m))|^q\,d\delta\right)^{\frac{1}{q}}$$
$$\leq\frac{\Omega(\flat_2,\flat_1,m)}{2}\left(\int_0^1|(1-2\delta)|^{\alpha p}\,d\delta\right)^{\frac{1}{p}}\left(\int_0^1((1-\delta)\,|\mathcal{F}'(m\flat_1)|^q+\delta\,|\mathcal{F}'(\flat_2)|^q)\right)^{\frac{1}{q}}$$
$$=\frac{\Omega(\flat_2,\flat_1,m)}{2(\alpha p+1)}^{\frac{1}{p}}\left(\frac{|\mathcal{F}'(m\flat_1)|^q+|\mathcal{F}'(\flat_2)|^q}{2}\right)^{\frac{1}{q}}.$$

This completes the proof. □

5 Application to special means:

We recall the following means for two real numbers $\flat_1,\flat_2,\flat_1\neq\flat_2$.

$$A(\flat_1,\flat_2)=\frac{\flat_1+\flat_2}{2},\flat_1,\flat_2\in R,$$

$$H(\flat_1,\flat_2)=\frac{2}{\frac{1}{\flat_1}+\frac{1}{\flat_2}},\flat_1,\flat_2\in R\backslash\{0\},$$

$$L(\flat_1,\flat_2)=\frac{\flat_2-\flat_1}{ln|\flat_2|-ln|\flat_1|},|\flat_1|\neq|\flat_2|,\flat_1\flat_2\in R,\flat_1\flat_2\neq 0,$$

$$L_n(\flat_1,\flat_2)=\left[\frac{\flat_2^{n+1}+\flat_1^{n+1}}{(n+1)(\flat_2-\flat_1)}\right]^{\frac{1}{n}},n\in Z\backslash\{-1,0\},\flat_1,\flat_2\in R,\flat_1\neq\flat_2.$$

Proposition 5.1. *Let $m\flat_1,m\flat_1+\Omega(\flat_2,\flat_1,m)\in R^+,m\flat_1<m\flat_1+\Omega(\flat_2,\flat_1,m)$. Then*

$$\left|A(m\flat_1^n,(m\flat_1+\Omega(\flat_2,\flat_1,m))^n)-L_n^n(m\flat_1,(m\flat_1+\Omega(\flat_2,\flat_1,m))\right|\leq\frac{n\Omega(\flat_2,\flat_1,m)}{2(p+1)^{\frac{1}{p}}}\left(\frac{m\flat_1^{(n-1)q}-\flat_2^{(n-1)q}}{2}\right)^{\frac{1}{q}}.$$

Proof. Applying Theorem 4.1 with $\mathcal{F}(w) = w^n, \Psi(w) = w, \alpha = 1$. Then we compute the result easily. □

Proposition 5.2. *Let* $m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m) \in R^+, m\flat_1 < m\flat_1 + \Omega(\flat_2, \flat_1, m)$. *Then*

$$|A(e^{m\flat_1}, e^{m\flat_1+\Omega(\flat_2,\flat_1,m)}) - L(e^{m\flat_1}, e^{m\flat_1+\Omega(\flat_2,\flat_1,m)})| \leq \frac{\Omega(\flat_2, \flat_1, m)}{2(p+1)^{\frac{1}{p}}} \left(\frac{e^{\flat_1 q} + e^{\flat_2 q}}{2} \right)^{\frac{1}{q}}.$$

Proof. Applying Theorem 4.1 with $\mathcal{F}(w) = e^w, \Psi(w) = w, \alpha - 1$. Then we compute the result easily. □

Proposition 5.3. *Let* $m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m) \in R^+, m\flat_1 < m\flat_1 + \Omega(\flat_2, \flat_1, m)$. *Then*

$$\left|H^{-1}(m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m)) - L^{-1}(m\flat_1, m\flat_1 + \Omega(\flat_2, \flat_1, m))\right| \leq \frac{\Omega(\flat_2, \flat_1, m)}{2(p+1)^{\frac{1}{p}}} \left[\frac{1}{2} \left(\frac{1}{\flat_1^{2q}} + \frac{1}{\flat_2^{2q}} \right) \right].$$

Proof. Applying Theorem 4.1 with $\mathcal{F}(w) = \frac{1}{w}$, $\psi(w) = w, \alpha = 1$. Then we compute the result easily. □

6 Conclusions

Fractional calculus has sparked the interest of multiple authors as well as scholars from a wide range of fields. Convexity theory allows us to create new, innovative numerical model frameworks that may be used to tackle a wide range of problems in the pure and applied sciences. Thus, convex analysis and its associated inequalities are growing in academic attention and appeal due to several advancements, modifications, and applications.

In this work:

(1) First, we investigated a new sort of H–H inequality via Ψ-RLFIO with some remarks and corollaries.

(2) We introduced a new lemma. Further, we discussed new refinement of H–H inequality based on newly constructed lemma.

(3) We introduced mean type applications in the frame of the Ψ-RLFIO .

This work contains intriguing methods and useful ideas that may be used to analyze Raina functions. We may talk about the above inequalities in the context of quantum calculus and interval analysis. One of the areas of research that is expanding the fastest is integral inequality. The application of interval-valued analysis and other forms of quantum calculus to integral inequalities should fascinate every scientist.

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