Research Articles

Global Journal of Sciences

Global Journal of Sciences
Year 2024, Volume-1, Issue-1 (January - June)

www.acspublisher.com

A Semi-Analytical Framework for higher-Order Delay Differential Equations: Utilizing Optimal Auxiliary Functions

Seema Bibi.[1], Rashid Nawaz[1], Laiq Zada[1], Negar. Alam[2], Nicholas Fewster-Young[3] ,

[1] Department of Mathematics, Abdul Wali Khan University Mardan, KP Pakistan.

[2] My school Sakhakot Education That Works, Pakistan.

[3]UniSA STEM, University of South Australia, Australia

[2]Corresponding author; email: rashid.ahmad@education.vic.gov.au

Abstract

Delay differential equations (DDEs) are extensively utilized in fields such as control systems, biology, and engineering to model processes where current states depend on past states, effectively accounting for time lags. Key applications include population dynamics, epidemic modeling, and economic systems, where delayed responses significantly influence system behavior. This paper presents the first extension of the Optimal Auxiliary Functions Method (OAFM) to second-order and third-order DDEs. The strength of this method lies in its convergence control parameters and auxiliary functions. Notably, the OAFM guarantees the convergence of approximate solutions after just one iteration, without requiring assumptions about small or large parameters. The method demonstrates both effectiveness and efficiency, with its accuracy validated through graphical and numerical results. Additionally, the results obtained are compared with those from the least squares method. Auxiliary functions and convergence control parameters are employed to further manage the convergence of the OAFM.

Key words: Delay differential equations, Least Square Method, Collocation Method, Optimal Auxiliary Functions Method.

1. Introduction

Many relevant investigations in the areas of physics, engineering, biomathematics, and others are mathematically modeled using delay differential equations (DDEs). DDEs are differential equations in which the derivatives of certain unknown functions at two different time instants are correlated (the past and the present). Researchers in the engineering and bioscience fields commonly come across mathematical models based on DDEs [1].

In the modern era, Minorsky was the first researcher to explore the following type of delay differential equations

$$\zeta'(\chi)=f\left(\chi,\zeta(\chi),\zeta(\chi-\tau)\right). \tag{1.1}$$

Typically, DDEs have been handled using discretization-based numerical techniques. This stems from these foundational techniques being suitably applied to solving first-order linear and simple non-linear DDEs. Due to their innately complex structure, DDEs are very difficult to study and if its possible to achieve analytical solutions are achieveable then they are implicit in nature[1].

The reasons for studying DDEs is to introduce naturally appearing delays in the systems whichgives models more life and realistic portrayal. This mathematically means DDEs have values which are dependent on previous solutions. Furthermore, time delays could be constant, dependent on time or state, or both[2].

The Delay Differential Equations (DDEs) were initially proposed in the 18^{th} century by Laplace and Condorcet [3]. However, the theory and applications of those equations did not start to develop quickly until after World War II, and they have continued ever since. In 1942, Pontryagin developed the fundamental theory governing the stability of systems defined by these kinds of equations. Smith published an important book in 1957, Pinny in 1958, Bellman and Cooke in 1963, Halanay in 1966, Myshkis in 1972, Hale in 1977, Yanusherski in 1978, and Marshal in 1979 all wrote significant publications [4].

To further motivate the study of DDEs, there are many physical and technical systems which include intrinsic delays,also known as heredity or memories, retarded arguments after actions, dead times, or time lags [5]. It may be challenging to include time delays in mathematical models because pure delays are frequently employed to illustrate the impact of transmission, transportation, and initial phenomena [6]. Therefore, theprovide an effective model for a wide range of phenomena in the applied sciences, including population dynamics, infectious disease, physiological and pharmaceutical kinetics, chemical kinetics, models of conveyor belts, urban traffic management, heat exchangers, robotics, navigational control of ships and aircraft, control theory, mathematical biology, mathematical economics, biochemical, medical, control system, biological models, and more general phenomena [7-9]. Delay equations used by Bunsen Berg et al. [10] for modeling the embryonic cell cycle. Patel et al. [11] proposed an iterative scheme for the optimal control systems with a quadratic cost functional.By using DDEs, the input-to-state stability of a time-invariant system with numerous non-commensurate and dispersed time delays as well as HIV-1 therapy for combating one virus with another, both have been recently modeled [7,12]. DDEs become particularly important when the model based on ordinary differential equations fails.

There are a few classes of nonlinear ODEs for which solutions are easy to find despite the obvious connections between ODE and DDEs. In contrast, there may be a number of distinct and significant ways in which the solutions to DDEs problems and those to ODE problems diverge [14]. There is always a vast range of frequencies produced by delay problems. Numerical techniques, asymptotic solutions, approximations, and graphical approaches are employed to solve them. Due to a significant increase in the use of delay models, several authors have examined and proposed numerous methods for solving DDEs, Variation Iteration Method (VIM) [14], Spline methods [15], Optimal Auxiliary Function Method [16-17], Homotopy Analysis Method (HAM) [18], Homotopy Perturbation Method (HPM) [19], Adomain Decomposition Method [20], Kudryashov's method [21], Modified Variational Iteration Algorithm-II [22], Iterative Decomposition Method [23], the Differential Transform Method [24], the Runge-Kutta Method [25], the Hermite Interpolation Method [26], the Variable Multistage Method [27], the Direct Block One Step Method [28], B-spline Collection Method [29], the Direct Two And Three-Point One-Step Block Method [30] etc.

The numerical solution of DDEs is extremely intriguing, and many techniques have been used to solve particular equations.. To demonstrate the analytical solution of homogeneous DDEs, Asl et al. [31] used Lambert functions. Also, Bellen et al. [32] described techniques for Gaussian points based on the predictor-corrector version of the one-step collocation method for non-stiff DDEs with time-dependent delays. Ismail et al. [33] compared the numerical results based on Newton Divided Difference and In't Hout interpolations, in order to solve delay differential equations. Martin et al. [34] introduced variable step size multistep methods. Evans et al. [35] used the

Adomian decomposition method and proposed a numerical method for linear and non-linear Higher Order Delay Differential Equations. While, Taiwo et al. [36] used an elementary decomposition methodfor solving Delay Differential Equations. In the scenario involving Constant and Variable Coefficients, Olvera et al. [37] expanded the enhanced the Multistage Homotopy perturbation method (EMHPM).

A series of authors have used transformation methods to solve the problem, the Differential Transform Method by Liu et al. for Delay Differential Equations [38], with Shampine et al. [39] suggested a numerical solution as well. Aboodh et al. [40] used the Aboodh Transformation method, Ebimene et al. [41] applied Elzaki Transformation Technique, and YAMAN et al. [42] used Daftardar-Jafari Method for solving Nonlinear Delay Differential Equations. Finally, the Sumudu Transform method (STM) was used to solve ageneric form of delay differential equations of the pantograph type [43].

In the literature, there have been many alternative techniques used as well for BVPs such as HPM by Bellen and Aslamnoor et al. [44] for DDE BVPs. The Laplace Adomian Decomposition Method (LADM) [45] for an second order of DDE BVP by Kanth et al.. More recently, Anakira et al. [46] expanded the applicability of the Optimal Homotopy Asymptotic Method (OHAM), in order to obtain the approximate analytic solution of DDEs. The dynamics of cutting machine operations were modeled by the stability lobes of DDEs, on the other hand, were determined by Insperger et al. [47] using the semi-discretization method. Based on the characteristics of the Chebyshev polynomials, Butcher et al. [48] created a method to obtain the stability lobes of milling machine operations and they demonstrated that this method is quicker than the full and semidiscretization methods because these solution techniques are approximations to the original DDEs by a series of ODEs [49]. To solve Delay Differential Equations, Adomian Decomposition Method (ADM) was utilized by Blanco et al. [50], Homotopy Perturbation Method (HPM) was used by Biazer et al. [51], the Homotopy Analysis Method (HAM) were studied by Alomari et al. [52]. In addition to particular types of equations, Predictor-corrector methods were examined by Bhalekar et al. [53]. For pantograph DDEs, the Residual Power Series Method was used by the authors of [54]. In [56- 58], there are many more numerical methods such as Galerkin and DDEs applications are discussed.

Stability, existence, and uniqueness of DDEs have been initially addressed in the works of [59-61]. The existence and uniqueness of DDEs were examined by Eloe et al. [62] andRebenda et al. [63] solves DDEs by the extension of semi-analytical technique. The method of DTM was used by Mohammed et al. [64], Mirzaee et al. [65] and Rostam et al. [66] to get at a numerical solution to DDEs. Verleydeu presented the collection method with an iterative linear system solver in 2003 [67] in order to compute the solution of a system of autonomous Delay Differential Equations. By using the direct Ritz method, Ordinary Delay Differential Equations were generalized to Partial Delay Differential Equations in 2004 and solved the variational formulation of the specific forms of partial delay differential equations [68]. In 2007, Luo [69] studied the exponential stability of nth order Delay Differential Equations. In 2006, Forde, in his doctoral dissertation [70] examined the modelling and stability of some biological systems as Delay Differential Equations. In 2001, Caus V. et al. [71] using non polynomial spline functions investigated the numerical stability and convergence of the numerical solution of Delay Differential Equations. Using the method of steps and the Laplace Transformation Method in 2009, Nagy T.K. studied the solution and stability of Delay Differential Equations [72].

In summary, delay differential equations (DDEs) play a crucial role in modeling real-world systems where current states are influenced by past states, capturing the dynamics of processes across various fields such as engineering, biology, and economics. Despite their significance, there remains a gap in the application of efficient and robust analytical methods for solving second-order and third-order DDEs. The Optimal Auxiliary Functions Method (OAFM) is introduced to address this gap, providing a systematic approach to obtain solutions that can enhance understanding and predictability in systems characterized by time delays.

Novelty

- **First-Time Application**: This paper presents the first application of the Optimal Auxiliary Functions Method (OAFM) to second-order and third-order Delay Differential Equations, marking a significant advancement in the methodology available for analyzing these equations. To the best of the author's knowledge, no one has yet applied the Optimal Auxiliary Functions Method (OAFM) to solve delay differential equations in the literature.

- **Expanded Research Base**: The method's successful application to a range of mathematical problems, such as Partial Differential Equations and the SEIR epidemic model, highlights its versatility and effectiveness in addressing both linear and nonlinear dynamics.

In Section 2, the Basic Idea of OAFM is briefly reviewed, and in Section 3, the governing equations are examined by addressing the approximations to the solutions of the second and third order delay differential equations. Section 4 also includes results and discussions. Finally, Section 5 discusses the conclusion.

2. Basic Idea of OAFM

The general form of nonlinear differential equation is given below:

$$L(\zeta(\chi))+\mathrm{N}(\zeta(\chi))+\mathrm{G}(\chi) = 0, \tag{1}$$

in which an unknown function is $\zeta(\chi)$, a linear operator is L, a nonlinear operator N, and a source operator G. The initial or boundary conditions are given below

$$\mathrm{B}\left(\zeta(\chi),\frac{d\zeta(\chi)}{d\chi}\right)=0. \tag{2}$$

We require an approximate solution $\tilde{\zeta}(\chi)$ for Eq. (1) and Eq. (2), with only two components:

$$\tilde{\zeta}(\chi) = \zeta_0(\chi) + \zeta_1(\chi ,\mathrm{C}_i),\ \ i=1,2,...,n \tag{3}$$

Where C_i from $i=1,2,...,n$ are currently unknown parameters. Putting Eq. (3) into Eq. (1), we get

$$L[\zeta_0(\chi)]+\mathrm{L}[\zeta_1(\chi,C_i)]\ +\mathrm{N}[\zeta_0(\chi)+\zeta_1(\chi,C_i)]+\mathrm{G}(\chi)=0. \tag{4}$$

To find the initial approximation $\zeta_0(\chi),$ the linear equation can be used

$$L[\zeta_0(\chi)]+\mathrm{G}(\chi)=0, \qquad \mathrm{B}\left(\zeta_0(\chi),\frac{d\zeta_0(\chi)}{d\chi}\right)=0 \tag{5}$$

The first approximation, yields from Eq. (5), the following equation

$$L[\zeta_1(\chi,C_i)]\ +\mathrm{N}[\zeta_0(\chi)+\zeta_1(\chi,C_i)]=0, \quad B\left(\zeta_1(\chi_i),\frac{d\zeta_1(\chi)}{d\chi}\right)=0. \tag{6}$$

In general, Eq. (6) is a nonlinear differential equation that is hard to solve. The nonlinear term from Eq. (6) is build up into the form at this point. Notice that the second term can be estimated as

$$\mathrm{N}\left[\zeta_0(\chi)+\zeta_1(\chi\ ,\mathrm{C}_i)\right]=\mathrm{N}\left[\zeta_0(\chi)\right]+\sum_{k=1}^{n}\frac{\zeta_1^k(\chi\ ,\mathrm{C}_i)}{k!}\mathrm{N}^{(k)}\left[\xi_0(\chi)\right], \tag{7}$$

where $n\to\infty$ *and* $\mathrm{N}^{(k)}=\dfrac{d^k\mathrm{N}}{d\chi^k}$. To resolve the difficulties \ arise in solving the nonlinear differential equation (6) from using equation (7) to accumulate the convergence of the first approximate solution $\tilde{\zeta}(\chi\ ,\mathrm{C}_i)$, we represent eq. (6) with an alternative form of eq. (7)

$$N[\zeta_0(\chi)+\zeta_1(\chi,C_i)]=A_1\big(\zeta_0(\chi),C_j\big)F\big[N\big(\zeta_0(\chi)\big)\big]+A_2(\zeta_0(\chi),C_k)$$

resulting in

$$L\left[\zeta_1(\chi,C_i)\right]+\mathrm{A}_1\left(\zeta_0(\chi),C_j\right)F\left[\mathrm{N}\left(\zeta_0(\chi)\right)\right]+\mathrm{A}_2\left(\zeta_0(\chi),C_k\right)=0,\ B\left(\zeta_1(\chi),\frac{d\zeta_1(\chi)}{d\chi}\right)=0 \tag{8}$$

where A_1 and A_2 are resultory auxiliary functions, and are selected based on the initial approximation $\zeta_0(\chi)$, or $\mathrm{N}\left[\zeta_0(\chi)\right]$, or in a combination of $\zeta_0(\chi)$ and $\mathrm{N}\left[\zeta_0(\chi)\right]$. The C_j and C_k, respectively are various unknown parameters with $j=1,2,..,\mathrm{p},\ k=p+1,p+2,...,n,\ i=j+k.$ and $F\left[\mathrm{N}\left(\zeta_0(\chi)\right)\right]$ is the operator component of $\mathrm{N}\left[\zeta_0(\chi)\right]$.

Minimizing the square residual error is one of the techniques that can be used to determine the unknown parameters C_j and C_k as accurately as possible

$$J\left(C_i,C_k\right)=\int_{(D)}R^2\left(\chi,C_i,C_k\right)d\chi, \tag{9}$$

where $R\left(\chi,C_i,C_k\right)=L\left[\tilde{\zeta}_1(\chi,C_i)\right]+\mathrm{N}\left[\tilde{\zeta}(\chi,C_i)\right]+\mathrm{G}\left(\chi\right),\ i=j+k,\ j=1,2,...,p,\ k=p+1,...,n.$

The residual minimization is subject to the following conditions:

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=...=\frac{\partial J}{\partial C_n}=0\,. \tag{10}$$

The convergence-control parameters can also be obtained by using the Ritz method, Galerkin method, Kantowich method, and collocation methods, etc.

Remark: The auxiliary functions can exist in either form $\zeta_0(\chi)$ or form $\mathrm{N}\left[\zeta_0(\chi)\right]$, or they can combine both forms.

- If $\zeta_0(\chi)$ or $\mathrm{N}[\zeta_0(\chi)]$ a polynomial function then the auxiliary functions should be the sum of polynomial functions.
- The auxiliary functions should be the sum of polynomial functions if $\zeta_0(\chi)$ or $\mathrm{N}[\zeta_0(\chi)]$ is a polynomial function.
- If $\zeta_0(\chi)$ or $\mathrm{N}[\zeta_0(\chi)]$ are trigonometric, the auxiliary functions should equal the sum of the trigonometric functions.

3. Governing Equations

To demonstrate the effectiveness and precision of the suggested method, we offer approximate solutions for second order and third order delay differential equations in this section. *Mathematica 10 is used to perform all computations.*

Example 1. Take the second order delay differential equations, for instance [36]

$$\frac{d^2\zeta}{d\chi^2}=\frac{3}{4}\zeta(\chi)+\zeta\left(\frac{\chi}{2}\right)-\chi^2+2,\quad 0\le\chi\le 1, \tag{14}$$

where the given initial condition is

$$\zeta(0)=0,\ \zeta'(0)=0. \tag{15}$$

The exact solution to equation (14) is given in [36], which is

$$\zeta(\chi)=\chi^2. \tag{16}$$

From eq. (14), linear and nonlinear expressions are given

$$\begin{cases} L(\zeta(\chi))=\dfrac{d^2\zeta}{d\chi^2}, \\ N(\zeta(\chi))=-\dfrac{3}{4}\zeta(\chi)+\zeta\left(\dfrac{\chi}{2}\right), \\ G(\chi)=\chi^2-2. \end{cases} \tag{17}$$

We obtain the following zero order problem by applying the OAFM described in section (2):

$$\frac{d^2\zeta_0}{d\chi^2}-\chi^2+2=0,\ \ \zeta_0(0)=0,\ \ \zeta_0'(0)=0. \tag{18}$$

The solution for eq. (18) is given,

$$\zeta_0(\chi)=\frac{1}{12}\left(12\chi^2+\chi^4\right). \tag{19}$$

If eq. (19) is substituted for the nonlinear part of eq. (17), we obtain

$$N(\zeta_0)=-\chi^2-\frac{13\chi^4}{192}. \tag{20}$$

By using OAFM, the first order problem is,

$$\frac{d^2\zeta_1}{d\chi^2}-\mathrm{A}_1N(\zeta_0)+\mathrm{A}_2=0,\quad \zeta_0(0)=0,\quad \zeta_0^{'}(0)=0, \tag{21}$$

We select the auxiliary functions A_1, A_2 in the following way,

$$\mathrm{A}_1=C_1\left(-\chi^2-\frac{13\chi^4}{192}\right)+C_2\left(-\chi^2-\frac{13\chi^4}{192}\right)^2+C_3\left(-\chi^2-\frac{13\chi^4}{192}\right)^4+C_4\left(-\chi^2-\frac{13\chi^4}{192}\right)^6, \tag{22}$$
$$\mathrm{A}_2=0.$$

The solution to eq. (21) is obtained by substituting eq. (20) and (22) into eq (21);

$$\zeta_1(\chi,C_i)=\left(\begin{array}{l}\frac{C_1\chi^6}{30}+\frac{(-13C_1+96C_2)\chi^8}{5376}-\frac{13(13C_1-576C_2)\chi^{10}}{3317760}+\\ \frac{(169C_2+12288C_3)\chi^{12}}{1622016}+\frac{(169C_2+184320C_3)\chi^{14}}{99090432}+\\ \frac{(845C_3+18432C_4)\chi^{16}}{4423680}+\frac{13(845C_3+129024C_4)\chi^{18}}{1082916864}+\\ \frac{169(845C_3+774144C_4)\chi^{20}}{516402708480}+\frac{2197(169C_3+1290240C_4)\chi^{22}}{120544699613184}\\ +\frac{999635C_4\chi^{24}}{750142881792}+\frac{199927C_4\chi^{26}}{4348654387200}+\frac{4826809C_4\chi^{28}}{5410421842378752}+\\ \frac{62748517C_4\chi^{30}}{8368119116212469760}\end{array}\right). \tag{23}$$

By combining eq. (19) and eq. (21), the first order approximate solution by OAFM can be found (23),

$$\tilde{\zeta}(\chi)=\zeta_0(\chi)+\zeta_1(\chi,C_1,C_2,C_3,C_4) \tag{24}$$

$$\tilde{\zeta}(\chi,C_i)=\left(\begin{array}{l}\frac{1}{12}\left(12\chi^2+\chi^4\right)+\frac{C_1\chi^6}{30}+\frac{\left(-13C_1+96C_2\right)\chi^8}{5376}-\frac{13\left(13C_1-576C_2\right)\chi^{10}}{3317760}+\\ \frac{\left(169C_2+12288C_3\right)\chi^{12}}{1622016}+\frac{\left(169C_2+184320C_3\right)\chi^{14}}{99090432}+\frac{\left(845C_3+18432C_4\right)\chi^{16}}{4423680}\\ +\frac{13\left(845C_3+129024C_4\right)\chi^{18}}{1082916864}+\frac{169\left(845C_3+774144C_4\right)\chi^{20}}{516402708480}+\frac{999635C_4\chi^{24}}{750142881792}\\ +\frac{2197\left(169C_3+1290240C_4\right)\chi^{22}}{120544699613184}-\frac{199927C_4\chi^{26}}{4348654387200}+\frac{4826809C_4\chi^{28}}{5410421842378752}+\\ \frac{62748517C_4\chi^{30}}{8368119116212469760}\end{array}\right). \tag{25}$$

Therefore, to calculate the exact values of the convergence control parameters C_i, $i=1,2,3...$, use the least squares method, as stated in eq. (25)

$$\begin{array}{l}C_1=7.43380999587875,\ C_2=12.046912067424115,\\ C_3=-9.2979628507085,\ C_4=3.5379687045401496.\end{array} \tag{26}$$

The first order OAFM solution is given as, after substitution of eq. (26) in eq. (25),

$$\tilde{\zeta}(\chi,C_i)=\left(\begin{array}{l}\chi^2(1.+0.08333333333333333\chi^2-0.24779366652929166\chi^4+0.19714732\\ 67348012\chi^6+0.026810547981640703\chi^8-0.06918392874676438\chi^{10}-0\\ .017274771639942003\chi^{12}+0.01296546326887011\chi^{14}+0.005385583543\\ 590115\chi^{16}+0.0008937711425761287\chi^{18}+0.00008316812359553707\chi^{20}\\ +0.0000047146716069801 28\chi^{22}+1.626561704408144\times10^{-7}\chi^{24}+3.1563\\ 34142197792\times10^{-9}\chi^{26}+2.65295326607142\times10^{-11}\chi^{28})\end{array}\right).$$

Table 1: Absolute error comparison for Problem 1 between the OAFM and the decomposition method.

χ	**OAFM Solution**	**Exact Solution.**	**Abs Errors [36]**	**Abs Error OAFM**
0.	0.	0.	0.	0.
0.1	0.0100	0.01	2.6909×10^{-5}	8.0875×10^{-6}
0.2	0.0401	0.04	1.0763×10^{-4}	1.1798×10^{-4}

0.3	0.0905	0.09	2.4218×10^{-4}	5.0741×10^{-4}
0.4	0.1612	0.16	4.3056×10^{-4}	1.2491×10^{-3}
0.5	0.2521	0.25	6.7277×10^{-4}	2.1151×10^{-3}
0.6	0.3625	0.36	9.6890×10^{-4}	2.5524×10^{-3}
0.7	0.4919	0.49	1.3191×10^{-3}	1.9559×10^{-3}
0.8	0.6400	0.64	1.7237×10^{-3}	8.9172×10^{-5}
0.9	0.8070	0.81	2.1837×10^{-3}	2.9623×10^{-3}
1	0.9923	1.	2.7003×10^{-3}	7.6282×10^{-3}

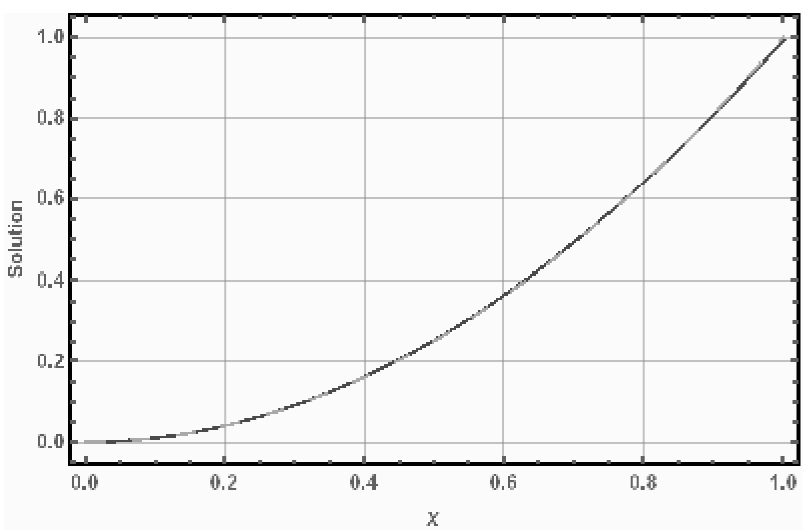

Figure 1. The 2D graph produced by the exact verses OAFM solution to Problem 1.

Example 2

Consider the third order delay differential equation [36]

$$\frac{d^3\zeta}{d\chi^3} = -1 + 2\zeta^2\left(\frac{\chi}{2}\right), \quad 0 \le \chi \le 1, \tag{27}$$

subject to the initial condition

$$\zeta_0(0) = 0, \ \zeta_0'(0) = 1, \ \zeta_0''(0) = 0. \tag{28}$$

The exact solution to equation (27) is given in [36], which is

$$\xi(\chi) = sin(\chi). \tag{29}$$

From eq. (27), the linear and nonlinear expressions are

$$\begin{cases} L(\zeta(\chi)) = \dfrac{d^3\zeta}{d\chi^3}, \\ N(\zeta(\chi)) = -2\zeta^2\left(\dfrac{\chi}{2}\right), \\ G(\chi) = 1. \end{cases} \tag{30}$$

Using the OAFM stated in section (2), we arrive at the following zero-order problem:

$$\zeta_0(\chi) = \frac{1}{6}\left(6\chi - \chi^3\right). \tag{31}$$

Here, we choose A_1, A_2 based on the first operator's non-linear operator.

$$\begin{aligned} A_1 &= C_1\left(\frac{\chi^4}{8}\right) + C_2\left(\frac{\chi^4}{8}\right)^2, \\ A_2 &= C_3\left(\frac{\chi^4}{8}\right)^4 + C_4\left(\frac{\chi^4}{8}\right)^6. \end{aligned} \tag{32}$$

We obtain the first-order solution by applying the OAFM method outlined for Problem 1:

$$\zeta_1(\chi)=\frac{1}{114422199091200}\begin{pmatrix}14189260800C_1\chi^9-601968640C_1\chi^{11}+7235200C_1\chi^{13}\\+520934400C_2\chi^{13}-27287040C_2\chi^{15}+380380C_2\chi^{17}-\\4804800C_3\chi^{19}-24871C_4\chi^{27}\end{pmatrix}. \tag{33}$$

We combine Eqs. (31) and (33) to get the OAFM solution of the first order:

$$\tilde{\zeta}(\chi,C_i)=\begin{pmatrix}\chi-\frac{\chi^3}{6}+\frac{C_1\chi^9}{8064}-\frac{C_1\chi^{11}}{190080}+\frac{(C_1+72\text{ C2})\,\chi^{13}}{15814656}-\frac{C_2\chi^{15}}{4193280}\\+\frac{C_2\chi^{17}}{300810240}-\frac{C_3\chi^{19}}{23814144}-\frac{C_4\chi^{27}}{4600627200}\end{pmatrix}. \tag{34}$$

The least squares method is used to determine the convergence control parameters found in equation (34). Eq. (35) provides the numerical values. These numbers in eq. (34) give us the first-order approximation of the answer to problem 2, shown in fig. 2.

$$\begin{aligned}&C_1=-64437.9111676810,\ C_2=3339289.387826368,\\&C_3=2.433012058446148,\ C_4=-1.30654767236302.\end{aligned} \tag{35}$$

Table 2: Third order delay differential equations for Problem 2, approximate solution found by OAFM.

χ	**OAFM Sol**	**Exact Sol [36]**	**Abs Error OAFM**
0.	0.	0.	0.
0.1	0.0998	0.0998	9.1299×10^{-8}
0.2	0.1986	0.1986	6.7360×10^{-6}
0.3	0.2951	0.2955	1.7447×10^{-4}
0.4	0.3821	0.3894	2.0646×10^{-3}
0.5	0.4655	0.4794	1.3888×10^{-2}
0.6	0.43216	0.56464	0.13247×10^{-1}

Table 3: Absolute error for the different numbers of the convergence control parameter obtained by OAFM.

χ	**OAFM Solution**	**Exact Solution**	**Absolute Error using** C_1 and C_2	**Absolute Error using** C_1, C_2 and C_3	**Absolute Error using** C_1, C_2, C_3 and C_4
0.	0.	0.	0.	0.	0.
0.1	0.09982	0.09983	6.02843×10^{-6}	1.35998×10^{-7}	9.12993×10^{-8}
0.2	0.19832	0.19866	3.42256×10^{-4}	1.60182×10^{-5}	6.73605×10^{-6}
0.3	0.29209	0.29552	3.42351×10^{-3}	$3.\ 53646\times10^{-4}$	1.74478×10^{-4}
0.4	0.37279	0.38941	1.66264×10^{-2}	3.24894×10^{-3}	2.06466×10^{-3}
0.5	0.42569	0.47942	5.37309×10^{-2}	1.75191×10^{-2}	1.38887×10^{-2}
0.6	0.43216	0.56464	0.13247×10^{-1}	6.49447×10^{-2}	6.10830×10^{-2}

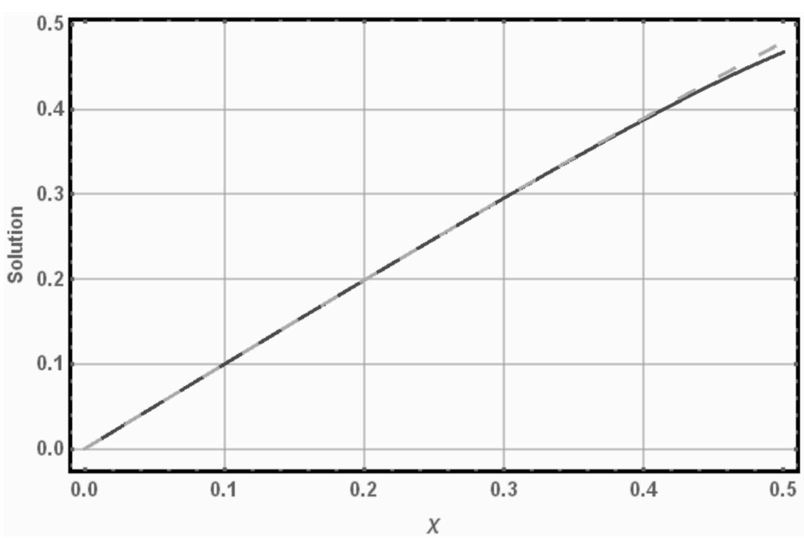

Figure 2. The 2D graph produced by the exact verses OAFM solution to Problem 2.

Example 3

Consider the second order Nonlinear proportional delay differential equation[42]

$$\frac{d^2\zeta}{d\chi^2} = -\zeta(\chi) + 5\zeta^2\left(\frac{\chi}{2}\right), \quad 0 \le \chi \le 1 \tag{36}$$

subject to the initial condition given by,

$$\zeta_0(0) = 1, \zeta_0^{'}(0) = -2. \tag{37}$$

The exact solution to eq. (36) is given in [42], which is

$$\zeta(\chi) = e^{-2\chi}. \tag{38}$$

The auxiliary functions A_1, A_2 can be choose for Example 3:

$$\begin{aligned} A_1 &= C_1(\cos(\chi)) + C_2(\cos(\chi))^2 + C_3(\cos(\chi))^4 + C_4(\cos(\chi))^6, \\ A_2 &= 0. \end{aligned} \tag{39}$$

then using the same procedure as discussed in Example 2, we get zero-order and the first order OAFM solution for Example 3:

$$\zeta_0(\chi) = \cos(\chi) - 2\sin(\chi). \tag{40}$$

$$\zeta_1(\chi, C_i) = \begin{pmatrix} \text{161.976 + (10.4752 - 51.2296 } \chi)\chi\text{+24.0056cos(2}\chi\text{)-1.71599 cos(3}\chi\text{)-} \\ \text{0.3777cos(4}\chi\text{)+0.0677949 cos(5}\chi\text{) + 0.0109339 cos(6}\chi\text{) -0.00240993} \\ \text{cos(7}\chi\text{) - 28.1586 sin(}\chi\text{) + cos(}\chi\text{) (-183.964+25.6664 sin(}\chi\text{)) -2.8251} \\ \text{5sin(3}\chi\text{) + 0.102989 sin(5}\chi\text{) - 0.00321323sin(7}\chi\text{)} \end{pmatrix}. \tag{41}$$

By using OAFM, the first order problem is obtained by combining eq. (40) and eq. (41),

$$\tilde{\zeta}(\chi, C_i) = \begin{pmatrix} (1/1128960)(-4821600\,C_2 -3732918\,C_3 -3147569\,C_4 + 840 \\ (\chi\ (-128\ (35\,C_2 +21\,C_3 +15\,C_4)+525(8\,C_2 +6\,C_3 +5\,C_4)\chi)- \\ 420\,C_1 (-37+8\chi+6\chi^2))-8820\ (1600\,C_1 -8\ (16+90\,C_2 +75\,C_3 \\ )-525\,C_4)\ \cos(\chi)+22050\ (48\,C_1 -80\ (C_2 +C_3)-75\,C_4)\cos(2 \\ \chi) +14700\ (16\,C_2 +20\,C_3 +21\,C_4)\cos(3\chi)\ -\ 55125\ (2\,C_3 +3 \\ C_4)\cos(4\chi)+5292\ (4\,C_3 +7\,C_4)\cos(5\chi)-12250\,C_4 \cos(6\chi)+ \\ 2700\,C_4 \cos(7\chi)\ -2257920\ \sin(\chi)+176400\ (16\,C_2 +8\,C_3 +5 \\ C_4)\sin(\chi)+1411200\,C_1\sin(2\chi)+19600(16\,C_2 +12\,C_3 +9\,C_4 \\ )\ \sin(3\chi)+\ 7056(4\,C_3 +5\,C_4)\sin(5\chi)+3600\,C_4 \sin(7\chi)) \end{pmatrix}. \tag{41}$$

The least squares method is used to determine the convergence control parameters found in equation (41). Eq.(41) provides the numerical values. These numbers in eq. (41) give us the first-order approximation of the answer to problem 2, shown in fig. 2.

$$\begin{aligned} C_1 &= 10.26654949107283, C_2 = -13.63809934533353, \\ C_3 &= 5.379148619033602, C_4 = -1.007669976790007. \end{aligned} \tag{42}$$

Table 4: Approximate solution, for nonlinear second order proportional Delay differential equations for Problem 3, obtained by OAFM.

χ	OAFM Solution	Exact solution [42]	Abs Error OAFM
0	1.	1.	1.1990×10^{-14}
0.1	0.8187	0.8187	1.2464×10^{-6}
0.2	0.6703	0.6703	2.9469×10^{-6}
0.3	0.5488	0.5488	4.6849×10^{-6}
0.4	0.4493	0.4493	6.4888×10^{-6}

0.5	0.3678	0.3678	8.9538×10^{-6}
0.6	0.3011	0.3011	1.4104×10^{-5}
0.7	0.2466	0.2465	1.7193×10^{-5}
0.8	0.2022	0.2018	3.3110×10^{-4}
0.9	0.1670	0.1652	1.7483×10^{-3}
1.	0.1415	0.1353	6.2440×10^{-3}

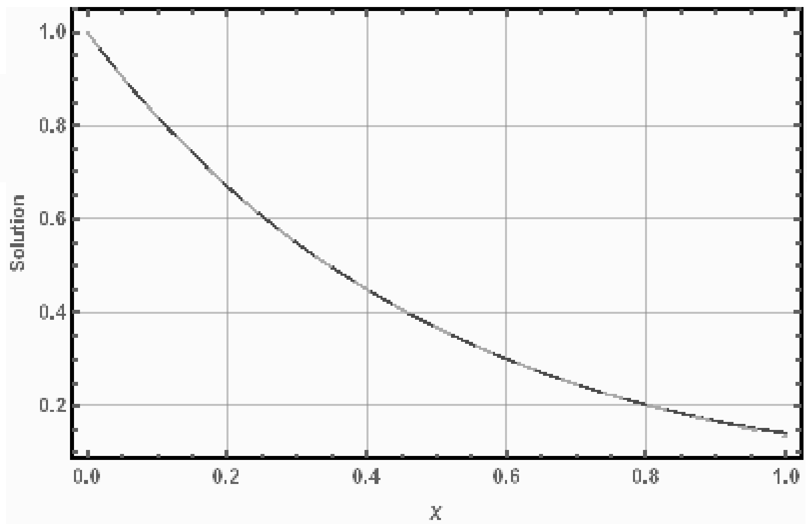

Figure 3. The 2D graph produced by the exact verses OAFM solution to Problem 3.

4. Results and discussions

Without the use of small parameter assumptions or discretization, the numerical problems of the formulation shown in section 3 and the extension of the (OAFM) scheme for (DDEs) presented in section 2 provide a highly accurate solution for the difficulties at hand, find the numerical approximation solutions to the second order and third order delay differential equations for problems 1-3. Figure 1 shows the approximate solution by OAFM for the exact solution $\zeta(\chi)$ and the second order approximate solution $\tilde{\zeta}(\chi, C_i)$. Comparisons of the absolute errors between the suggested method and the exact solution are shown in Table 1. The OAFM is seen to converge to exact solution. Figure 2 shows the approximate solution via OAFM for the exact solution $\zeta(\chi)$ and the third order approximate solution $\tilde{\zeta}(\chi, C_i)$. The approximate and exact solutions are shown in Table 2 respectively. Table 3 shows the Absolute error for the different numbers of the convergence control parameter obtained by OAFM, which shows that increasing the number of convergence control parameters, the OAFM converged rapidly to exact solution. The numerical values of the convergence control parameters are calculated using the least square method. Figure 3 shows the approximate solution via OAFM for the exact solution $\zeta(\chi)$ and the second order approximate solution $\tilde{\zeta}(\chi, C_i)$. The approximate and exact solutions are shown in Table 4 respectively. The OAFM is seen to converge to exact solution. The numerical values of the convergence control parameters are calculated using the collocation method.

5. Conclusion

The Optimal Auxiliary Functions Method (OAFM) has been applied for the first time to solve second-order and third-order delay differential equations without requiring assumptions about small or large parameters. This innovative approach provides a more generalizable solution framework, yielding highly accurate numerical approximations when compared to exact solutions. The OAFM demonstrates efficient convergence, achieving rapid convergence to the exact solution after just one iteration, which highlights its effectiveness. The method utilizes auxiliary functions and carefully selected convergence control parameters to ensure reliable convergence. Results are illustrated through figures and tables, which show the effectiveness of the OAFM by comparing approximate solutions with exact solutions, revealing low absolute errors. Moreover, the OAFM effectively addresses strong nonlinear problems, showcasing its robustness compared to traditional methods. Notably, the OAFM offers significant advantages over other perturbation and numerical methods by eliminating the need for discretization, simplifying the implementation process. Increasing the number of convergence control parameters further enhances the accuracy and speed of convergence to the exact solution. The numerical values of these convergence control parameters are calculated using both collocation and least squares methods, providing a comprehensive analysis of the approach.

References

[1] Mohsen, L., 2016. *The Classification of Delay Differential Equations Using Lie Symmetry Analysis* (Doctoral dissertation, University Technology Malaysia).

[2] Aibinu, M.O., Thakur, S.C. and Moyo, S., 2021. Solving delay differential equations via Sumudu transform. *arXiv preprint arXiv:2106.03515.*

[3] Gorecki, H., Fuksa, S., Grabowski, P. and Korytowski, A., 1989. Analysis and synthesis of time delay systems.

[4] Asl, F.M. and Ulsoy, A.G., 2003. Analysis of a system of linear delay differential equations. *J. Dyn. Sys., Meas., Control*, *125*(2), pp.215-223.

[5] Niculescu, S.I., 2001. *Delay effects on stability: a robust control approach* (Vol. 269). Springer Science & Business Media.

[6] Asl, F.M. and Ulsoy, A.G., 2003. Analysis of a system of linear delay differential equations. *J. Dyn. Sys., Meas., Control*, *125*(2), pp.215-223.

[7] Lv, C. and Yuan, Z., 2009. Stability analysis of delay differential equation models of HIV-1 therapy for fighting a virus with another virus. *Journal of Mathematical Analysis and Applications*, *352*(2), pp.672-683.

[8] Marchuk, G.I., 1997. *Mathematical modelling of immune response in infectious diseases* (Vol. 395). Springer Science & Business Media.

[9] Glass, L. and Mackey, M.C., 1979. Pathological conditions resulting from instabilities in physiological control systems. *Annals of the New York Academy of Sciences*, *316*(1), pp.214-235.

[10] Busenberg, S. and Tang, B., 1994. Mathematical models of the early embryonic cell cycle: the role of MPF activation and cyclin degradation. *Journal of mathematical biology*, *32*, pp.573-596.

[11] Patel, N.K., Das, P.C. and Prabhu, S.S., 1982. Optimal control of systems described by delay differential equations. *International Journal of Control*, *36*(2), pp.303-311.

[12] Pepe, P., Karafyllis, I. and Jiang, Z.P., 2008. On the Liapunov–Krasovskii methodology for the ISS of systems described by coupled delay differential and difference equations. *Automatica*, *44*(9), pp.2266-2273.

[13] Baker, C.T., Paul, C.A. and Willé, D.R., 1995. Issues in the numerical solution of evolutionary delay differential equations. *Advances in Computational Mathematics*, *3*, pp.171-196.

[14] Rangkuti, Y.M. and Noorani, M.S.M., 2012. The exact solution of delay differential equations using coupling variational iteration with Taylor series and small term. *Bulletin of Mathematics*, *4*(01), pp.1-15.

[15] El-Safty, A. (1993) Approximate Solution of the Delay Differential Equation with Cubic O. M. Ogunlaran, A. S. Olagunju 674 Spline Functions. Bulletin of the Faculty Science, Assiut University, 22, 67-73

[16] Zada, L., Nawaz, R., Ayaz, M., Ahmad, H., Alrabaiah, H. and Chu, Y.M., 2021. New algorithm for the approximate solution of generalized seventh order Korteweg-Devries equation arising in shallow water waves. Results in Physics, 20, p.103744.

[17] Nawaz, R., Zada, L., Ullah, F., Ahmad, H., Ayaz, M., Ahmad, I. and Nofal, T.A., 2021. An extension of optimal auxiliary function method to fractional order high dimensional equations. Alexandria Engineering Journal, 60(5), pp.4809-4818.

[18] Alomari, A.K., Noorani, M.S.M. and Nazar, R., 2009. Solution of delay differential equation by means of homotopy analysis method. *Acta Applicandae Mathematicae*, *108*, pp.395-412.

[19] Shakeri, F. and Dehghan, M., 2008. Solution of delay differential equations via a homotopy perturbation method. *Mathematical and computer Modelling*, *48*(3-4), pp.486-498.

[20] Evans, D.J. and Raslan, K.R., 2005. The Adomian decomposition method for solving delay differential equation. *International Journal of Computer Mathematics*, *82*(1), pp.49-54.

[21] Yusuf, A., Sulaiman, T.A., Khalil, E.M., Bayram, M. and Ahmad, H., 2021. Construction of multi-wave complexiton solutions of the Kadomtsev-Petviashvili equation via two efficient analyzing techniques. Results in Physics, 21, p.103775.

[22] Ahmad, H., Khan, T.A., Stanimirović, P.S., Chu, Y.M. and Ahmad, I., 2020. Modified Variational Iteration Algorithm-II: Convergence and Applications to Diffusion Models. *Complexity*, *2020*(1), p.8841718.

[23] Mirzaee, F. and Latifi, L., 2011. Numerical solution of delay differential equations by differential transform method.

[24] Ismail, F., Al-Khasawneh, R.A. and Suleiman, M., 2002. Numerical treatment of delay differential equations by Runge-Kutta method using Hermite interpolation. *MATEMATIKA: Malaysian Journal of Industrial and Applied Mathematics*, pp.79-90.

[25] Oberle, H.J. and Pesch, H.J., 1981. Numerical treatment of delay differential equations by Hermite interpolation. *Numerische Mathematik*, *37*, pp.235-255.

[26] Martin, J.A. and García, O., 2002. Variable multistep methods for higher-order delay differential equations. *Mathematical and computer modelling*, *36*(7-8), pp.805-820.

[27] Abdul Majid, Z., Mokhtar, N.Z. and Suleiman, M., 2012. Direct two-point block one-step method for solving general second-order ordinary differential equations. *Mathematical Problems in Engineering*, *2012*.

[28] Kumar, D. and Kadalbajoo, M., 2012. Numerical treatment of singularly perturbed delay differential equations using B-Spline collocation method on Shishkin mesh. *Jnaiam*, *7*(3-4), pp.73-90.

[29] Rasdi, N., Majid, N.A., Ismail, F., Senu, N., See, P.P. and Radzi, H.M., 2013. Solving second order delay differential equations by direct two- and three-point one-step block method.

[30] Asl, F.M. and Ulsoy, A.G., 2000, June. Analytical solution of a system of homogeneous delay differential equations via the Lambert function. In *Proceedings of the 2000 American Control Conference. ACC (IEEE Cat. No. 00CH36334)* (Vol. 4, pp. 2496-2500). IEEE.

[31] Bellen, A. and Zennaro, M., 1985. Numerical solution of delay differential equations by uniform corrections to an implicit Runge-Kutta method. *Numerische Mathematik*, *47*, pp.301-316.

[32] Ismail, F., Al-Khasawneh, R.A. and Suleiman, M., 2003. Comparison of interpolations used in solving delay differential equations by Runge-Kutta method. *International journal of computer mathematics*, *80*(7), pp.921-930.

[33] Martin, J.A. and García, O., 2002. Variable multistep methods for higher-order delay differential equations. *Mathematical and computer modelling*, *36*(7-8), pp.805-820.

[34] Evans, D.J. and Raslan, K.R., 2005. The Adomian decomposition method for solving delay differential equation. *International Journal of Computer Mathematics*, *82*(1), pp.49-54.

[35] Taiwo, O.A. and Odetunde, O.S., 2010. On the numerical approximation of delay differential equations by a decomposition method. *Asian Journal of Mathematics and Statistics*, *3*(4), pp.237-243.

[36] Olvera, D., Elías-Zúñiga, A., López de Lacalle, L.N. and Rodríguez, C.A., 2015, January. Approximate solutions of delay differential equations with constant and variable coefficients by the enhanced multistage homotopy perturbation method. In *Abstract and Applied Analysis* (Vol. 2015). Hindawi.

[37] Liu, B., Zhou, X. and Du, Q., 2015. Differential transform method for some delay differential equations. *Applied Mathematics*, *6*(3), p.585.

[38] Shampine, L.F. and Thompson, S., 2009. Numerical solution of delay differential equations. *Delay Differential Equations: Recent Advances and New Directions*, pp.1-27.

[39] Aboodh, K.S., Farah, R.A., Almardy, I.A. and Osman, A.K., 2018. Solving delay differential equations by Aboodh transformation method. *International Journal of Applied Mathematics & Statistical Sciences*, *7*(2), pp.55-64.

[40] Ebimene, J.M. and Njoseh. I.N. solving delay differential equation by Elazaki transform method.

[41] YAMAN, V. and YİLMAZ, B., Solutions of Nonlinear Delay Differential Equations by Daftardar-Jafari Method. *Turkish Journal of Mathematics and Computer Science*, *10*, pp.95-106.

[42] Aibinu, M.O., Thakur, S.C. and Moyo, S., 2021. Solving delay differential equations via Sumudu transform. *arXiv preprint arXiv:2106.03515.*

[43] Yousef, H.M. and Ismail, A.I.B.M.D., Homotopy Perturbation Method for Boundary Value Problems with Delay Differential Equations.

[44] Kiymaz, O., 2009. An algorithm for solving initial value problems using Laplace Adomian decomposition method.

[45] Ratib Anakira, N., Alomari, A.K. and Hashim, I., 2013. Optimal homotopy asymptotic method for solving delay differential equations. *Mathematical Problems in Engineering*, *2013*.

[46] Insperger, T. and Stépán, G., 2011. *Semi-discretization for time-delay systems: stability and engineering applications* (Vol. 178). Springer Science & Business Media.

[47] Butcher, E.A., Nindujarla, P. and Bueler, E., 2005, January. Stability of up-and down-milling using Chebyshev collocation method. In *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference* (Vol. 47438, pp. 841-850).

[48] Insperger, T., 2010. Full-discretization and semi-discretization for milling stability prediction: some comments. *International Journal of Machine Tools and Manufacture*, *50*(7), pp.658-662.

[49] Blanco-Cocom, L., Estrella, A.G. and Avila-Vales, E., 2012. Solving delay differential systems with history functions by the Adomian decomposition method. *Applied Mathematics and Computation*, *218*(10), pp.5994-6011.

[50] Biazar, J. and Ghanbari, B., 2012. The homotopy perturbation method for solving neutral functional–differential equations with proportional delays. *Journal of King Saud University-Science*, *24*(1), pp.33-37.

[51] Alomari, A.K., Noorani, M.S.M. and Nazar, R., 2009. Solution of delay differential equation by means of homotopy analysis method. *Acta Applicandae Mathematicae*, *108*, pp.395-412.

[52] Bhalekar, S.A.C.H.I.N. and Daftardar-Gejji, V.A.R.S.H.A., 2011. A predictor-corrector scheme for solving nonlinear delay differential equations of fractional order. *J. Fract. Calc. Appl*, *1*(5), pp.1-9.

[53] Komashynska, I., Al-Smadi, M., Al-Habahbeh, A. and Ateiwi, A., 2016. Analytical approximate solutions of systems of multi-pantograph delay differential equations using residual power-series method. *arXiv preprint arXiv:1611.05485*.

[54] Liu, B. and Zhang, C., 2015. A spectral galerkin method for nonlinear delay convection–diffusion–reaction equations. *Computers & Mathematics with Applications*, *69*(8), pp.709-724.

[55] Bellen, A. and Zennaro, M., 2013. *Numerical methods for delay differential equations*. Oxford university press.

[56] Rebenda, J. and Šmarda, Z., 2017. A differential transformation approach for solving functional differential equations with multiple delays. *Communications in Nonlinear Science and Numerical Simulation*, *48*, pp.246-257.

[57] In't Hout, K.J., 2001. Convergence of Runge-Kutta methods for delay differential equations. *BIT Numerical Mathematics*, *41*, pp.322-344.

[58] Benhammouda, B., Vazquez-Leal, H. and Hernandez-Martinez, L., 2014. Procedure for exact solutions of nonlinear pantograph delay differential equations. *British Journal of Mathematics & Computer Science*, *4*(19), pp.2738-2751.

[59] Ding, L., Li, X. and Li, Z., 2010. Fixed points and stability in nonlinear equations with variable delays. *Fixed Point Theory and Applications*, *2010*, pp.1-14.

[60] Karakoç, F.A.T.M.A. and Bereketoğlu, H., 2009. Solutions of delay differential equations by using differential transform method. *International Journal of Computer Mathematics*, *86*(5), pp.914-923.

[61] Eloe, P.W., Raffoul, Y.N. and Tisdell, C.C., 2005. Existence, uniqueness and constructive results for delay differential equations. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, *2005*, pp.Paper-No.

[62] Rebenda, J., Šmarda, Z. and Khan, Y., 2015. A new semi-analytical approach for numerical solving of Cauchy problem for functional differential equations. *arXiv preprint arXiv:1501.00411*.

[63] Mohammed, G.J. and Fadhel, F.S., 2017. Extend differential transform methods for solving differential equations with multiple delay. *Ibn AL-Haitham Journal for Pure and Applied Science*, *24*(3).

[64] Mirzaee, F. and Latifi, L., 2011. Numerical solution of delay differential equations by differential transform method.

[65] Saeed, R.K. and Rahman, B.M., 2011. Differential transform method for solving system of delay differential equation. *Australian Journal of Basic and Applied Sciences*, *5*(4), pp.201-206.

[66] Verheyden, K. and Lust, K., 2005. A Newton-Picard collocation method for periodic solutions of delay differential equations. *BIT Numerical Mathematics*, *45*, pp.605-625.

[67] H.M. Binte, "Variational formulation of partial delay differential equations," Ph.D. thesis, Al-Mustansiriyah University, 2004.

[68] Luo, J., 2007. A note on exponential stability in pth mean of solutions of stochastic delay differential equations. *Journal of computational and applied mathematics*, *198*(1), pp.143-148.

[69] Forde, J.E., 2005. *Delay differential equation models in mathematical biology*. University of Michigan.

[70] Caus, V.A. and Micula, G., 2001. Numerical solution of the delay differential equations by non-polynomial spline functions. *Studia University Babes-Bolyai, Informatica*, *46*, pp.91-98.

[71] Kalmár-Nagy, T., 2009. Stability analysis of delay-differential equations by the method of steps and inverse Laplace transform. *Differential Equations and Dynamical Systems*, *17*, pp.185-200.